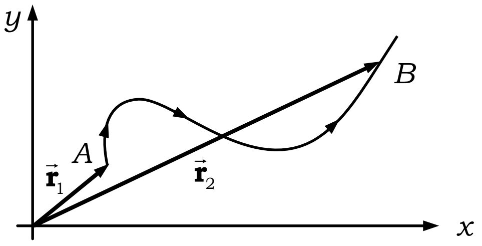
MOTION IN ONE & TWO DIMENSION

Reference frame :

Is the co-ordinate system with respect to which the position, displacement, velocity acceleration etc. of a particle are defined is called frame of reference. In our universe there exist no absolute frame of reference. Every known thing is moving with some velocity or the other. Until specifically mentioned we will consider the frame of reference associated with the earth as the absolute frame of reference.

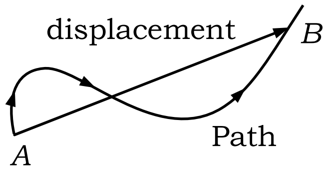
Position :

Position of a point is represented by a vector starting form origin of frame of reference and ending at the point whose position is represented by this vector. In the adjoining diagram the position vectors of points A and B are shown by $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ respectively.



Displacement :

Change in position vector is defined as displacement vector.



$$\overrightarrow{AB} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = \vec{\mathbf{s}}\,.$$

Thus the shortest distance (straight line distance) from initial position A to final position B of an object is the magnitude of displacement. Displacement is a vector quantity and it is in the direction from initial position to final position. In the adjacent figure $\overrightarrow{AB}$ represents the displacement.

Distance :

Distance between two points is defined as the length of path from one point to the other. As there can exist more than one path between two points the distance between same pair of points can be different along different paths. The straight-line distance (the shortest possible distance) between two points gives the magnitude of displacement as one moves from one point to the other. This discussion makes it clear that the displacement from one point to the other and the distance (l) between same pair of points are related as

$$l \geq |\vec{\mathbf{s}}|$$

Average velocity :

If the total displacement of an object is $\Delta\vec{\mathbf{r}} = \vec{\mathbf{s}}$ and total time of journey is Δt then average velocity of the object is defined as $\Delta\vec{\mathbf{s}}$ divided by Δt.

$$\langle \vec{\mathbf{v}} \rangle = \frac{\Delta \vec{\mathbf{r}}}{\Delta t}$$

Average speed :

It is defined as the total distance travelled divided by total time of journey

$$\langle v \rangle = {}^{l}\!/_{t}$$

As the distance travelled is always greater than or at least equal to the magnitude of displacement in the same interval of time between same pair of points hence the average speed in a particular interval of time is greater than or at least equal to the magnitude of average velocity in the same interval of time.

$$\langle v \rangle \geq \left| \langle \vec{\mathbf{v}} \rangle \right|$$

Instantaneous velocity :

The rate of displacement of an object is called instantaneous velocity. In fact instantaneous velocity of the object at a particular instant is same as average velocity of the object in a infinitesimally small duration of time around that particular instant.

$$\vec{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{r}}}{\Delta t} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d\vec{\mathbf{s}}}{dt}$$

Instantaneous speed :

The rate at which the distance is travelled at a particular instant is called instantaneous speed. It is the average sped of the object in a infinitesimally small duration of time around that particular instant.

$$v = \lim_{\Delta t \to 0} \frac{\Delta l}{\Delta t} = \frac{dl}{dt}$$

As the distance travelled in a very small duration of time is also very small and the curve of very small length can be approximated to a straight-line segment, hence the instantaneous speed is same as the magnitude of the instantaneous velocity at that instant.

$$v = \left| \vec{\mathbf{v}} \right|$$

Average acceleration :

Average acceleration of the object is defined as the change in velocity divided by the time interval during which the change in velocity occurred

$$\langle \vec{\mathbf{a}} \rangle = \frac{\Delta \vec{\mathbf{v}}}{\Delta t} = \frac{\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i}{\Delta t}$$

Instantaneous Acceleration :

The rate of change of velocity of the object is called its instantaneous acceleration. Similar to instantaneous velocity the instantaneous acceleration of the object at a particular instant is same as average acceleration of the object in a infinitesimally small duration at that particular instant.

$$\vec{\mathbf{a}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{v}}}{\Delta t} = \frac{d\vec{\mathbf{v}}}{dt} \quad \text{therefore}$$

$$\vec{\mathbf{a}} = \frac{d^2\vec{\mathbf{s}}}{dt^2} = \frac{d\vec{\mathbf{v}}}{dt} = v\frac{d\vec{\mathbf{v}}}{ds}$$

Fundamental Relations :

Average velocity

$$\langle \vec{\mathbf{v}} \rangle = \frac{\vec{\mathbf{s}}}{t} \qquad \ldots\ldots \text{(a)}$$

Instantaneous velocity

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{s}}}{dt} \qquad \ldots\ldots \text{(b)}$$

instantaneous acceleration

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} \quad \text{or} \quad \vec{\mathbf{a}} = v\frac{d\vec{\mathbf{v}}}{ds} \qquad \ldots\ldots \text{(c)}$$

average acceleration

$$\langle \vec{\mathbf{a}} \rangle = \frac{\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i}{t} \qquad \ldots\ldots \text{(d)}$$

The above four relations are fundamental relations and valid for any kind of translatory motion of a point.

Equations of Motion :

A complete description of the motion of a particle can be obtained if we know the mathematical dependence of its position x on time t for all times. i.e.,

$$x = f(t)$$

For the constant acceleration of a particle

$$a = \frac{dv}{dt} = \text{constant}$$

$$\Rightarrow \quad dv = adt$$

$$\Rightarrow \quad \int_u^v dv = a\int_0^t dt$$

$$\Rightarrow \quad v - u = a(t - 0)$$

$$\Rightarrow \quad v = u + at$$

Now, the instantaneous velocity of the particle is given as

$$v = \frac{dx}{dt}$$

$$\Rightarrow \quad dx = vdt = (u + at)dt$$

$$\Rightarrow \quad \int_{x_0}^x dx = \int_0^t (u + at)dt$$

$$\Rightarrow \quad x - x_0 = s = ut + \frac{1}{2}at^2$$

Now, the instantaneous acceleration of the particle is given as

$$a = v\frac{dv}{dx}$$

$$\Rightarrow \quad vdv = adx$$

$$\Rightarrow \quad \int_u^v vdv = a\int_{x_0}^x dx$$

$$\Rightarrow \quad \frac{1}{2}v^2 - \frac{1}{2}u^2 = a\left(x - x_0\right) = as$$

$$\Rightarrow \quad v^2 = u^2 + 2as$$

From the definition the average velocity

$$\langle v\rangle = \frac{s}{t}$$

$$\Rightarrow \qquad \langle v\rangle = \frac{ut + \left(1/2\right)at^2}{t}$$

$$\Rightarrow \qquad \langle v\rangle = \frac{2u + at}{2}$$

$$\Rightarrow \qquad \langle v\rangle = \frac{u + v}{2}$$

The distance travelled in n^{th} second

$$s_{n^{th}} = s_n - s_{(n-1)}$$

$$\Rightarrow \qquad s_{n^{th}} = \left\{un + \frac{1}{2}an^2\right\} - \left\{u(n-1) + \frac{1}{2}a(n-1)^2\right\}$$

$$\Rightarrow \qquad s_{n^{th}} = u + \frac{1}{2}a(2n-1)$$

Therefore the following five relations

$$v = u + at \qquad \text{.... (1)}$$

$$s = ut + \frac{1}{2}at^2 \qquad \text{.... (2)}$$

$$v^2 = u^2 + 2as \qquad \text{.... (3)}$$

$$\langle v\rangle = \frac{u+v}{2} \qquad \text{.... (4)}$$

$$s_{n^{th}} = u + \frac{1}{2}a(2n-1) \qquad \text{.... (5)}$$

are called equations of motion. These are **valid only in case of motion with uniform acceleration**. The above relations are basically vector equations but when used for motion in one dimension, either to or fro direction is taken positive and according to this convention sine of various quantities are decided. Generally the direction of final displacement is taken +ve.

The above equations of motion can also be applied for a point moving on a curved path with uniform tangential acceleration (the component of instantaneous acceleration along the tangent to the curve). In that case velocities become tangential velocities (speeds) and the displacement is replaced by length of curve travelled.

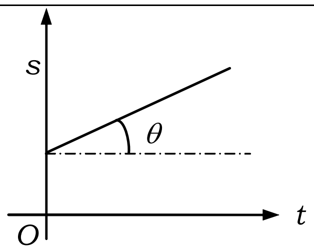
Graphical representation :

Displacement-time graph :

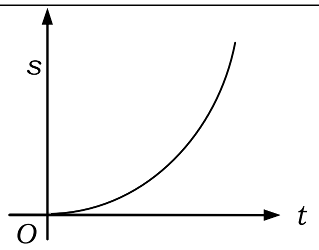
Displacement of the particle plotted against time is called displacement-time graph.

- ⇨ It gives instantaneous displacement at a particular instant
- ⇨ Slope of the tangent to the displacement-time curve at any instant gives instantaneous velocity at that instant.

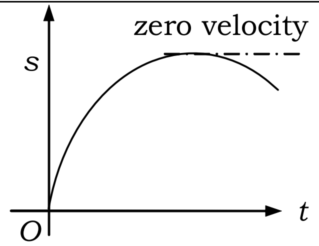
Therefore if the plot of the displacement-time graph is a straight line it shows the uniform velocity, concave curve shows increasing velocity (acceleration) as the slope increases and the convex curve shows decreasing velocity (deceleration).



uniform velocity
as the slope of s-t curve is constant

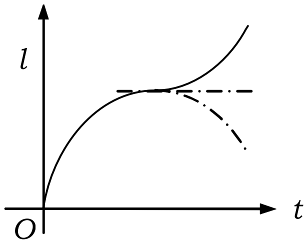


acceleration
as the slope of s-t curve is increasing



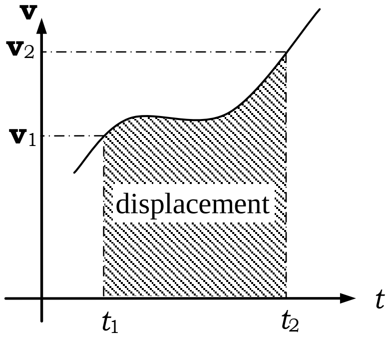
deceleration
as the slope of s-t curve is decreasing

Distance-time graph :
It is a monotonically increasing graph. Till magnitude of displacement is increasing with time the distance-time graph follows the displacement-time graph. When the magnitude of displacement in displacement-time graph starts decreasing the distance-time still increases and drawn as mirror image about a line parallel to time axis.



Slope of distance-time graph gives magnitude of instantaneous velocity or instantaneous speed of the particle.

Velocity-time graph :
Velocity of the particle plotted against time is called velocity-time graph. Following are the information provided by velocity-time graph



- ⇨ It gives instantaneous velocity at a particular instant
- ⇨ Slope of the tangent to the velocity-time curve at any instant gives instantaneous acceleration at the particular instant. Therefore if the plot of the velocity-time graph is a straight line it shows the uniform acceleration.

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \tan\theta$$

- ⇨ Area under the velocity-time graph in a particular duration gives total displacement of the object in that time interval.

$$\vec{\mathbf{s}} = \int_{t_1}^{t_2} \vec{\mathbf{v}}\,dt$$

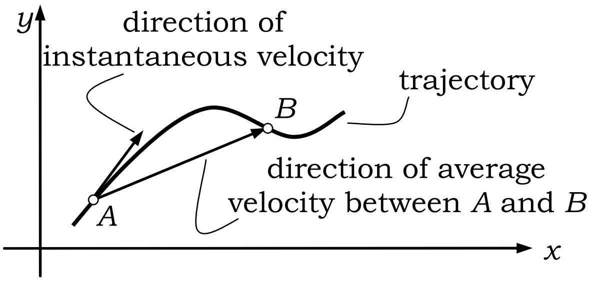
Acceleration-time graph :

Acceleration of the particle plotted against time is called acceleration-time graph. Area below the acceleration-time graph gives change in velocity.

x–y graph (the trajectory) :

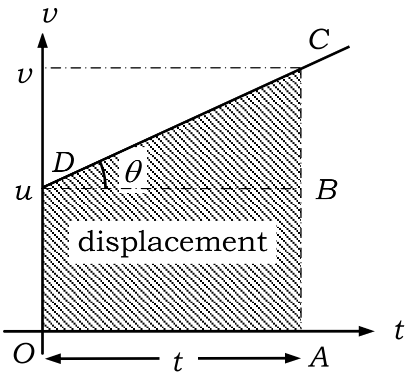
The *x–y* graph of a particle gives path on which particle is travelling, also called the trajectory of the particle.

- ⇨ Tangent to a point on the trajectory gives direction of motion of particle (the direction of instantaneous velocity) at that point.
- ⇨ The line joining one point on the trajectory to the other point on the trajectory gives direction of average velocity of particle while moving from one point to the other as it is the direction of displacement vector.



Equations of motion using velocity time graph :

Now let us consider the uniform acceleration. The velocity-time graph will be a straight line. The acceleration of the object is slope of the line *CD*



$$a = \tan\theta = \frac{BC}{BD} = \frac{v-u}{t}$$

$\Rightarrow \quad v = u + at$ .... (1)

The total displacement of the object is area *OABCD*

s = area $OABCD$

$s = \square\ OABD + \Delta\ BCD$

$s = OA \times OB + \frac{1}{2} BD \times BC$

$s = ut + \frac{1}{2}at^2$ .... (2)

Again s = area $OABCD$

$\Rightarrow \quad s = \frac{1}{2}(AC + OD) \times OA$

$$\Rightarrow \quad s = \frac{1}{2}(v+u)\times t$$

$$\Rightarrow \quad s = \frac{1}{2}(v+u)\times\frac{(v-u)}{a}$$

$$\Rightarrow \quad s = \frac{v^2-u^2}{2a} \qquad \text{\{from equation (1)\}}$$

$$v^2 = u^2 + 2as \qquad \ldots.\ (3)$$

Average velocity

$$\langle v\rangle = \frac{\text{area } OABCD}{OA}$$

$$\langle v\rangle = \frac{(OD+AC)\times OA}{2\times OA}$$

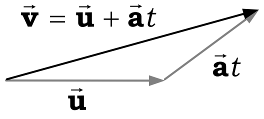
$$\langle v\rangle = \frac{u+v}{2} \qquad \ldots.\ (4)$$

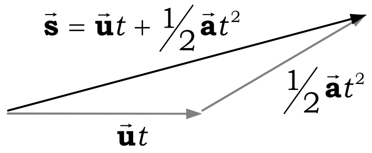
Vector form of equations of motion :

$$\vec{\mathbf{v}} = \vec{\mathbf{u}} + \vec{\mathbf{a}}t$$

$$\vec{\mathbf{s}} = \vec{\mathbf{u}}t + \frac{1}{2}\vec{\mathbf{a}}t^2$$

$$\vec{\mathbf{v}}.\vec{\mathbf{v}} = \vec{\mathbf{u}}.\vec{\mathbf{u}} + 2\vec{\mathbf{a}}.\vec{\mathbf{s}} \qquad \text{(here all the three terms are scalar)}$$

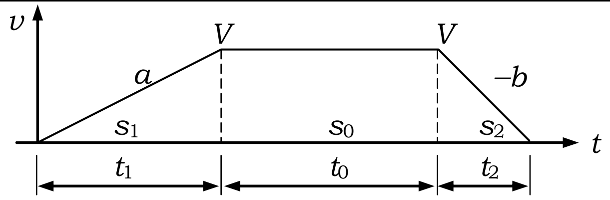




If initial velocity is zero, then the equation of motion can be written as: $v = at$ $s = \frac{1}{2}at^2$ $v^2 = 2as$ $\langle v\rangle = \frac{v}{2}$	The same is true if final velocity is zero and initial velocities is u. (thus then the acceleration = $-a$) $v = 0 = u - at$ ⇒ $u = at$ $s = ut - \frac{1}{2}at^2 = at^2 - \frac{1}{2}at^2$ ⇒ $s = \frac{1}{2}at^2$ $v^2 = 0 = u^2 - 2as$ ⇒ $u^2 = 2as$ $\langle v\rangle = \frac{0+u}{2} = \frac{u}{2}$ ⇒ $\langle v\rangle = \frac{u}{2}$

Thus the equations of motion have same shape whether the initial velocity is zero or the final velocity is zero.

To solve problems where the entire journey is covered in three parts, first starting from rest with uniform acceleration, second with uniform velocity, which is maximum in the entire journey and finally coming to rest with uniform deceleration, the velocity time graph is shown in the adjacent diagram. From the above results



$$V = at_1 = bt_2 \qquad .... (1)$$

$$\frac{V}{2} = \frac{s_1}{t_1} = \frac{s_2}{t_2} \qquad .... (2)$$

Total time of journey

$$\tau = t_1 + t_2 + t_0 = \frac{2s_1}{V} + \frac{2s_2}{V} + \frac{s_0}{V}$$

$$\tau = \frac{2}{V}\left(S - \frac{s_0}{2}\right) \qquad .... (3)$$

Total displacement

$$S = s_1 + s_2 + s_0 = \frac{Vt_1}{2} + \frac{Vt_2}{2} + Vt_0$$

$$S = \frac{V}{2}(\tau + t_0) \qquad .... (4)$$

We will need either equation (3) or (4) to solve every problem of this kind along with equation (1).
If the magnitude of acceleration and retardation are same ($a = b$) then necessarily $s_1 = s_2$ and $t_1 = t_2$.

Representative example 1 :

For the first $\left(1/m\right)$ *of the distance between two stations a train is uniformly accelerated starting from rest and for last* $\left(1/n\right)$ *of the distance it is uniformly retarded to come to rest again. Prove that the ratio of its greatest velocity to its average velocity in the entire journey is*

$$1 + \frac{1}{m} + \frac{1}{n} : 1$$

Solution :

Let total distance of the journey be s then from equation (3)

$$\tau = \frac{2\left(s/m\right)}{V} + \frac{2\left(s/n\right)}{V} + \frac{\left(s - s/m - s/n\right)}{V}$$

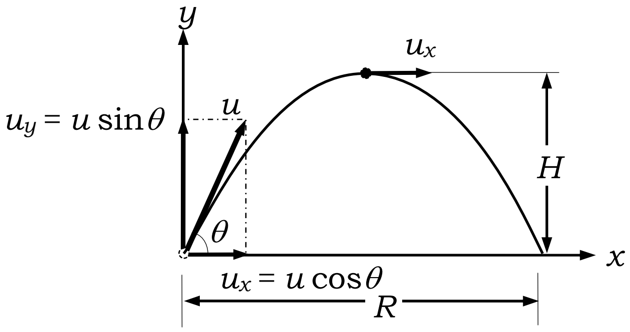
$$\Rightarrow \qquad \tau = \frac{s}{V}\left(1 + 1/m + 1/n\right)$$

$$\Rightarrow \qquad \frac{V}{\left(s/\tau\right)} = \frac{V}{\langle v\rangle} = \left(1 + 1/m + 1/n\right)$$

PROJECTILE MOTION :

Mutually perpendicular components of a motion are independent of each other. The only link between them is time.

If a particle is projected with a velocity of projection u, at an angle of projection θ made with the horizontal, it travels on a curved path as shown in the following diagram. This motion can be considered as the combination of motion along horizontal direction (x-axis) with uniform velocity as there is no component of acceleration due to gravity exists in the x-axis and the motion along vertical direction (y-axis) with down ward acceleration due to gravity g.



$u_x = u\cos\theta,\ a_x = 0$

and $u_y = u\sin\theta,\ a_y = -g$

writing equations of motion along x and y-axis respectively we get

$$x = u\cos\theta\, t \qquad \text{.... (1)}$$

$$y = u\sin\theta\, t - \frac{1}{2}gt^2 \qquad \text{.... (2)}$$

celiminating t from (1) and (2) we get

$$y = (\tan\theta)\, x - \left(\frac{g}{2u^2}\sec^2\theta\right)x^2 \qquad \text{.... (3)}$$

The above equation shows the relation between x and y and represents the path of the projectile known as its equation of trajectory. The inspection of equation (3) shows that the expression is the equation of a parabola of the form

$y = bx + cx^2$ where

$b = \tan\theta$, and

$$c = -\frac{g}{2u^2\cos^2\theta}.$$

Both a and c are constants for given velocity and angle of projection. Thus, the trajectory of a projectile is a parabola.

The projectile finally comes back to the horizontal plane (x-axis) after completing its motion, thus $y = 0$ at the end of the journey. The total time of journey is called time of flight of projectile and till then its horizontal displacement called range of projectile R. Putting this condition to equation (2) we get

$$0 = u\sin\theta\, t - \frac{1}{2}gt^2$$

$$\Rightarrow \quad t = 0,\ t = \frac{2u\sin\theta}{g}$$

The two values of t show that $y = 0$ at the time of projection and at

$$T = \frac{2u\sin\theta}{g} \qquad \text{.... (4)}$$

is called time of flight.

From equation (1), in this time the horizontal displacement is

$$R = u\cos\theta\,\frac{2u\sin\theta}{g}$$

$$\Rightarrow \qquad R = \frac{u^2 \sin 2\theta}{g} \qquad \ldots.\ (5)$$

is called range of projectile.

Since the projectile motion is symmetric about vertical line passing through its highest point thus maximum height attained by the projectile is its vertical displacement y in a time $\left(T/2\right)$, therefore from equation (2)

$$H = u \sin\theta \frac{u \sin\theta}{g} - \frac{1}{2} g \left(\frac{u \sin\theta}{g}\right)^2$$

$$H = \frac{u^2 \sin^2\theta}{2g} \qquad \ldots.\ (6)$$

is called height of projectile.

For maximum range the projectile should be projected with maximum velocity and at such an angle so that sin 2θ is maximum, thus sin $2\theta = 1 \Rightarrow \sin\left(\pi/2\right)$ therefore range R is maximum for an angle of projection $\left(\pi/4\right)$ and

$$R_{max} = \frac{u^2}{g}. \qquad \ldots.\ (7)$$

Similarly for maximum height the projectile should be projected with maximum velocity and at such an angle so that $\sin^2\theta$ is maximum, thus sin $\theta = 1 \Rightarrow \sin\left(\pi/2\right)$ therefore height H is maximum for an angle of projection $\left(\pi/2\right)$ and

$$H_{max} = \frac{u^2}{2g}. \qquad \ldots.\ (8)$$

For complimentary angle of projections range of projectile remains same if the velocity of projection remains same in both the cases.

$$R_{\left(\pi/2-\theta\right)} = \frac{u^2 \sin 2\left(\pi/2 - \theta\right)}{g}$$

$$\Rightarrow \qquad R_{\left(\pi/2-\theta\right)} = \frac{u^2 \sin\left(\pi - 2\theta\right)}{g}$$

$$\Rightarrow \qquad R_{\left(\pi/2-\theta\right)} = \frac{u^2 \sin 2\theta}{g}$$

$$\Rightarrow \qquad R_{\left(\pi/2-\theta\right)} = R_\theta$$

For same velocity of projection

$$R/H = \frac{2 \sin 2\theta}{\sin^2\theta} = 4 \cot\theta$$

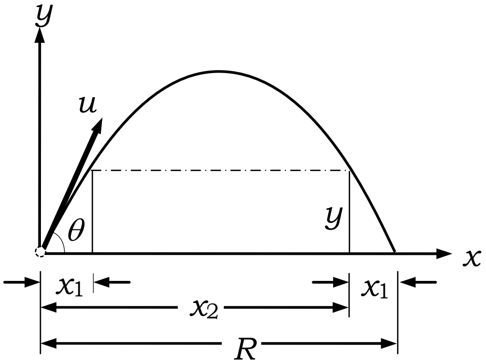
$$\Rightarrow \qquad 4H = R \tan\theta$$

The angle of projection for which height of projectile and its range are same is given by

$$\frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin^2\theta}{2g}$$

$$\Rightarrow \qquad \theta = \tan^{-1}(4)$$

Sum of two roots of equation of trajectory comes out to be equal to range of projectile.



$$\left(\frac{g}{2u^2}\sec^2\theta\right)x^2-(\tan\theta)\,x+y=0$$

$$x_1+x_2=\frac{\tan\theta}{\frac{g}{2u^2}\sec^2\theta}=\frac{u^2\sin 2\theta}{g}=R$$

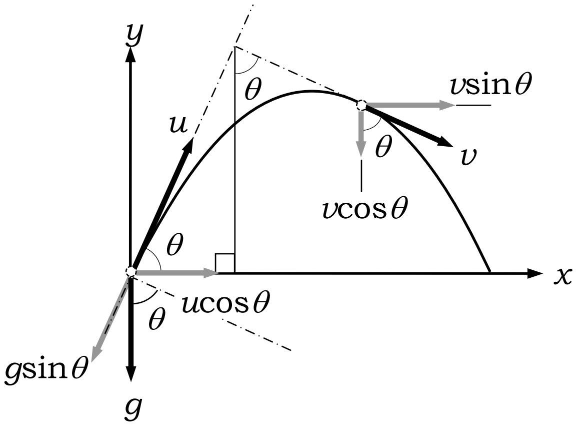
This underlines the fact that the two points at which height of the projectile from ground is same are symmetrically placed from both the ends of the trajectory of projectile on the horizontal plane.

Representative example 2 :

A particle is projected with a velocity u at an angle α from horizontal. After what time will it be travelling perpendicular to the initial direction of projection.

Solution :

As the horizontal component of velocity remains constant in a projectile motion, hence



$$v\sin\theta=u\cos\theta$$

$$\Rightarrow\qquad v=\frac{u\cos\theta}{\sin\theta}$$

from vertical component of motion

$$-v\cos\theta=u\sin\theta-gt$$

$$\Rightarrow\qquad gt=u\sin\theta+u\frac{\cos^2\theta}{\sin\theta}$$

$\Rightarrow \quad t = \dfrac{u}{g\sin\theta}$

or

if we consider the velocities as vectors

$$\vec{\mathbf{u}} = u\cos\theta\,\hat{\mathbf{i}} + u\sin\theta\,\hat{\mathbf{j}} \quad \text{and}$$

$$\vec{\mathbf{v}} = u\cos\theta\,\hat{\mathbf{i}} + (u\sin\theta - gt)\,\hat{\mathbf{j}}$$

for $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ to be perpendicular

$$\vec{\mathbf{u}}.\vec{\mathbf{v}} = 0$$

$\Rightarrow \quad \left(u\cos\theta\,\hat{\mathbf{i}} + u\sin\theta\,\hat{\mathbf{j}}\right).\left(u\cos\theta\,\hat{\mathbf{i}} + (u\sin\theta - gt)\,\hat{\mathbf{j}}\right) = 0$

$\Rightarrow \quad u^2\cos^2\theta + u^2\sin^2\theta - u\sin\theta\,gt = 0$

$\Rightarrow \quad t = \dfrac{u}{g\sin\theta}$

or

if we consider the velocity vector at time t as

$$\vec{\mathbf{v}} = \vec{\mathbf{u}} + \vec{\mathbf{g}}t$$

for $\vec{\mathrm{u}}$ and $\vec{\mathrm{v}}$ to be perpendicular

$$\vec{\mathbf{u}}.\vec{\mathbf{v}} = 0$$

$\Rightarrow \quad \vec{\mathbf{u}}.\left(\vec{\mathbf{u}} + \vec{\mathbf{g}}t\right) = 0$

$\Rightarrow \quad \vec{\mathbf{u}}.\vec{\mathbf{u}} + \vec{\mathbf{u}}.\vec{\mathbf{g}}t = 0$

$\Rightarrow \quad u^2 + ugt\cos\left(\pi/2 + \theta\right) = 0$

$\Rightarrow \quad t = \dfrac{u}{g\sin\theta}$

or

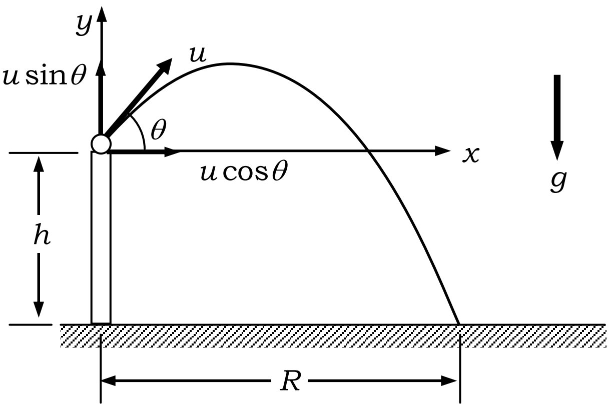
if we consider the direction of u as the reference direction then when the particle is travelling perpendicular to u velocity along u becomes zero and the acceleration along u is $g\sin\theta$, hence

$$v_{\text{along } u} = 0 = u - g\sin\theta\, t$$

$\Rightarrow \quad t = \dfrac{u}{g\sin\theta}$

Projectile from an elevated point :

When an object is projected from a point at a height h from the ground then



$$h = -u\sin\theta\, t + \tfrac{1}{2} g t^2$$

$$\Rightarrow \quad \left(\tfrac{g}{2}\right)t^2 - u\sin\theta\, t - h = 0 \quad \text{hence}$$

$$t = \frac{u\sin\theta \pm \sqrt{u^2\sin^2\theta + 2gh}}{g}$$

as $\sqrt{u^2\sin^2\theta + 2gh} > u\sin\theta$

and t can't be negative therefore

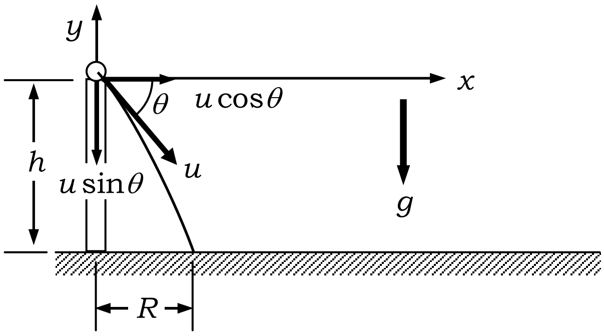
$$t = \frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g}$$

and

$$R = u\cos\theta\, t$$

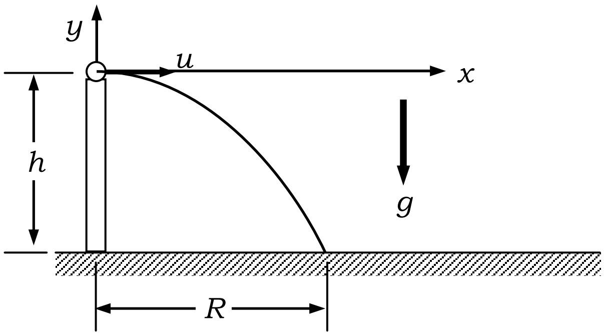
$$\Rightarrow \quad R = u\cos\theta\left\{\frac{u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g}\right\}$$

if the object is projected at an angle θ below the horizontal, θ is –ve.



$$t = \frac{-u\sin\theta + \sqrt{u^2\sin^2\theta + 2gh}}{g}$$

If the object is projected horizontally then $\theta = 0$, and



since initial vertical velocity is zero hence

$$t = \sqrt{\frac{2h}{g}}$$

therefore range on horizontal ground

$$R = ut$$

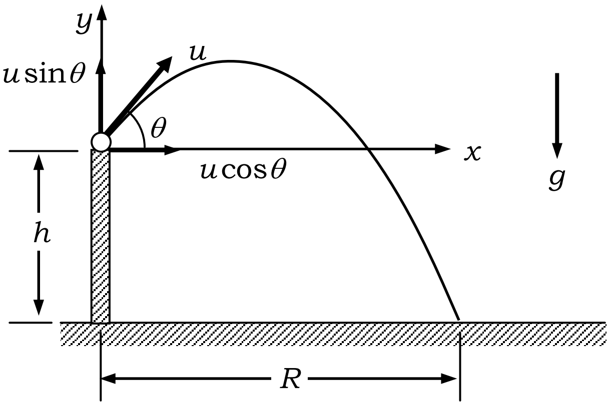
$\Rightarrow \quad R = u\sqrt{\dfrac{2h}{g}}$

Representative example 3 :

An object is projected at an angle θ from the horizontal from the top of a tower of height h with a velocity u. Calculate the angle of projection for which the range on the horizontal plane will be maximum. Also calculate the maximum range.

Solution :

From the following diagram



$$R = u\cos\theta\,\tau$$

$$-h = u\sin\theta\,\tau - \frac{1}{2}g\tau^2$$

where τ is total time of journey.

Eliminating τ we get

$$-h = R\tan\theta - \left(\frac{g}{2u^2}\sec^2\theta\right)R^2 \qquad \text{.... (1)}$$

For R to be maximum $\dfrac{dR}{d\theta} = 0$. Therefore differentiating (1) and putting $\dfrac{dR}{d\theta} = 0$.

$$0 = \left(R\sec^2\theta + \tan\theta\frac{dR}{d\theta}\right) - \left(\frac{g}{2u^2}\right)\left(2\sec^2\theta\tan\theta R^2 + 2R\frac{dR}{d\theta}\right)$$

$$\Rightarrow \quad R\sec^2\theta - \left(\frac{g}{2u^2}\right)2\sec^2\theta\tan\theta R^2 = 0$$

$$\Rightarrow \quad 1 - \frac{g}{u^2}\tan\theta R = 0$$

$$\Rightarrow \quad R = \frac{u^2}{g\tan\theta} = R_{max}$$

Putting this value of R to get corresponding value of θ

$$-h = \frac{u^2}{g} - \left(\frac{g}{2u^2}\sec^2\theta\right)\left(\frac{u^2}{g\tan\theta}\right)^2$$

$$\Rightarrow \quad h = \frac{u^2}{g}\left(\frac{1}{2\sin^2\theta} - 1\right)$$

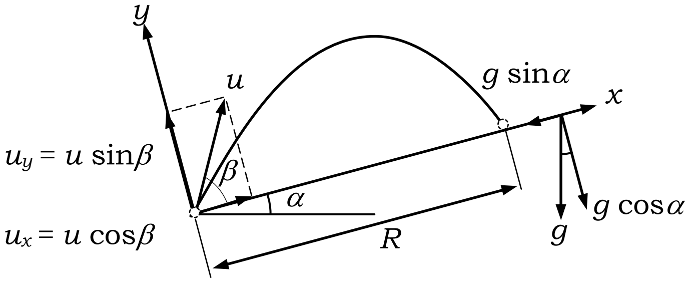
$$\Rightarrow \quad \theta = \sin^{-1}\frac{u}{\sqrt{2(u^2+gh)}} \quad \text{or}$$

$$\theta = \tan^{-1}\frac{u}{\sqrt{u^2+2gh}} \quad \text{and}$$

$$R_{\max} = \frac{u}{g}\sqrt{u^2+2gh}$$

Projectile on incline plane :

Let us assume direction along incline as x-axis and perpendicular to it as y-axis.



$u_x = u\cos\beta,\ a_x = -g\sin\alpha$

and $u_y = u\sin\beta,\ a_y = -g\cos\alpha$;

writing equations of motion along x and y-axis respectively we get

$$x = u\cos\beta\, t - \left(\tfrac{g}{2}\right)\sin\alpha\, t^2 \qquad \text{.... (9)}$$

$$y = u\sin\beta\, t - \left(\tfrac{g}{2}\right)\cos\alpha\, t^2 \qquad \text{.... (10)}$$

The projectile finally comes back to the incline plane (x-axis) after completing its motion, thus $y = 0$ at the end of the journey. Putting this condition in equation (10)

$$0 = u\sin\beta\, t - \left(\tfrac{g}{2}\right)\cos\alpha\, t^2$$

$\Rightarrow$ $t = 0,\ t = \dfrac{2u\sin\beta}{g\cos\alpha}$ therefore the time of flight is

$$T = \frac{2u\sin\beta}{g\cos\alpha} \qquad \text{.... (11)}$$

In this much time displacement along the incline plane (x-axis) is called range along the incline, thus from equation (9)

$$R = u\cos\beta\left(\frac{2u\sin\beta}{g\cos\alpha}\right) - \tfrac{1}{2}g\sin\alpha\left(\frac{2u\sin\beta}{g\cos\alpha}\right)^2$$

$$\Rightarrow \quad R = \frac{2u^2\sin\beta\cos(\alpha+\beta)}{g\cos^2\alpha}$$

The range along the incline remains same for the angle of projection β and $\left\{\tfrac{\pi}{2} - (\alpha+\beta)\right\}$ above the incline, keeping the velocity of projection same.

$$R_{\left\{\frac{\pi}{2}-(\alpha+\beta)\right\}} = \frac{2u^2\sin\left\{\tfrac{\pi}{2}-(\alpha+\beta)\right\}\cos\left\{\alpha+\left(\tfrac{\pi}{2}-\overline{\alpha+\beta}\right)\right\}}{g\cos^2\alpha}$$

$$\Rightarrow \quad R_{\{\pi/2-(\alpha+\beta)\}} = \frac{2u^2\sin\beta\cos(\alpha+\beta)}{g\cos^2\alpha} = R_\beta$$

Since R depends on the angle of projection β above the incline plane therefore for maximum range

$$\frac{dR}{d\beta} = \frac{2u^2}{g\cos^2\alpha}\frac{d}{d\beta}\{\sin\beta\cos(\alpha+\beta)\}$$

$$\Rightarrow \quad \frac{dR}{d\beta} = \frac{2u^2}{g\cos^2\alpha}\cos(\alpha+2\beta) = 0 \qquad \left(\text{for maxima/minima } \frac{dR}{d\beta} = 0\right)$$

$$\Rightarrow \quad \cos(\alpha+2\beta) = 0$$

$$\Rightarrow \quad \alpha+2\beta = \pi/2$$

From above discussion it is clear that the condition for maximum range along the plane is that it should be projected along the angular bisector of the angle between the vertical and incline plane. This condition is universally true. Whether the projectile is projected up the incline plane, down the incline plane or even for the projectile on horizontal plane where the range is maximum when angle of projection is $(\pi/4)$ for maximum range along horizontal plane, which is the angular bisector of vertical and horizontal plane, the plane along which the range should be maximum. Maximum range up the incline plane

$$R_{max} = \frac{2u^2\sin\beta\cos(\alpha+\beta)}{g\cos^2\alpha}$$

$$\Rightarrow \quad R_{max} = \frac{2u^2\sin^2\beta}{g\cos^2\alpha}$$

$$\Rightarrow \quad R_{max} = \frac{u^2(1-\cos 2\beta)}{g\cos^2\alpha}$$

$$\Rightarrow \quad R_{max} = \frac{u^2(1-\sin\alpha)}{g(1-\sin^2\alpha)}$$

$$\Rightarrow \quad R_{max} = \frac{u^2}{g(1+\sin\alpha)}$$

Similarly the maximum range along the incline when projectile is projected down the plane is

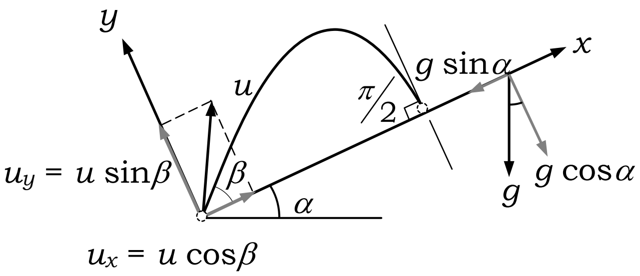
$$R_{max}' = \frac{u^2}{g(1-\sin\alpha)}$$

Condition for object to hit the incline perpendicular to it :

When the projectile hits the incline perpendicular to it, its velocity along the incline becomes zero. Therefore

$$v = 0 = u\cos\beta - g\sin\alpha\, t$$

$$\Rightarrow \quad t = \frac{u\cos\beta}{g\sin\alpha}$$



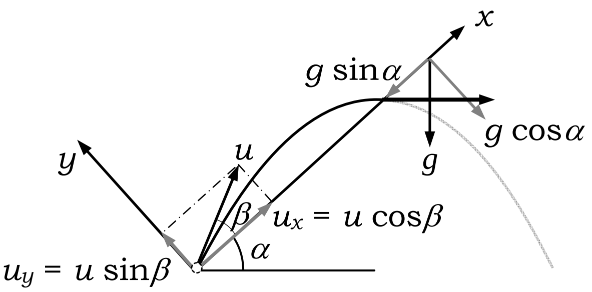
equating this time of flight with the above result

$$\frac{u\cos\beta}{g\sin\alpha} = \frac{2u\sin\beta}{g\cos\alpha}$$

$\Rightarrow$ $\tan\alpha\tan\beta = \frac{1}{2}$ is the desired condition.

<u>Condition for object to hit the incline plane horizontally</u> :

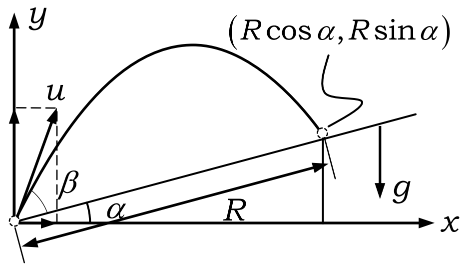
When the projectile hits the incline plane horizontally, it is at the highest point of its trajectory therefore its time of flight on incline is same as half of the time of flight on horizontal plane



$$\frac{2u\sin\beta}{g\cos\alpha} = \frac{1}{2}\frac{2u\sin(\alpha+\beta)}{g}$$

$$\Rightarrow \quad \frac{\sin\beta}{\cos\alpha\sin(\alpha+\beta)} = \frac{1}{2}$$

All of the above relations can also be obtained considering horizontal and vertical axis system.



With angle of projection being $(\alpha+\beta)$, putting the values in equation of trajectory with horizontal axes

$$R\sin\alpha = (R\cos\alpha)\tan(\alpha+\beta) - \frac{g}{2u^2}\sec^2(\alpha+\beta)(R\cos\alpha)^2$$

$$\Rightarrow \quad \frac{g}{2u^2}\sec^2(\alpha+\beta)R\cos^2\alpha = \cos\alpha\tan(\alpha+\beta)-\sin\alpha$$

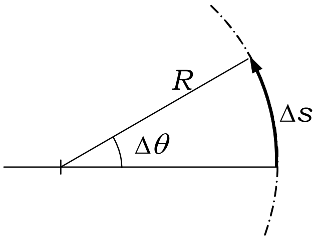
$$\Rightarrow \quad \frac{g}{2u^2}\sec^2(\alpha+\beta)R\cos^2\alpha = \frac{\sin\beta}{\cos(\alpha+\beta)}$$

$$\Rightarrow \quad R = \frac{2u^2\sin\beta\cos(\alpha+\beta)}{g\cos^2\alpha}$$

KINEMATICS OF CIRCULAR MOTION :

Angular displacement :

In magnitude it is equal to the angle subtended by the arc travelled by the object at the centre and its direction is perpendicular to the plane of rotation given by right hand screw law. i.e. for clockwise rotation the direction of angular displacement is going in to the plane of the paper ⊗ while for counter clockwise rotation it is coming out of the plane of the paper ⊙.



Angular displacement Is the angle subtended by arc travelled by object at the centre of circle. It is a vector quantity and in the direction of axis of rotation given by right hand rule.

$$\Delta\theta = \frac{\Delta s}{R} \qquad \text{.... (1)}$$

in vector terms

$$\Delta\vec{\boldsymbol{\theta}} = \frac{\vec{\mathbf{r}}\times\Delta\vec{\mathbf{s}}}{r^2}$$

Average angular velocity :

If the total angular displacement of an object is $\Delta\vec{\theta}$ and total time of journey is Δt then average velocity of the object is defined as the ratio of $\Delta\vec{\theta}$ and Δt.

$$\langle\vec{\omega}\rangle = \frac{\Delta\vec{\theta}}{\Delta t}$$

Instantaneous angular velocity :

The rate of angular displacement of an object is called instantaneous angular velocity. In fact instantaneous angular velocity of the object at a particular instant is same as average angular velocity of the object in a infinitesimally small duration at that particular instant.

$$\vec{\omega} = \lim_{\Delta t\to 0}\frac{\Delta\vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt}$$

Average angular acceleration :

Average angular acceleration of the object is defined as the change in angular velocity divided by the time interval during which the change in angular velocity occurred

$$\vec{\alpha} = \frac{\Delta\vec{\omega}}{\Delta t} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{\Delta t}$$

Instantaneous angular acceleration :

The rate of change of angular velocity of the object is called its instantaneous angular acceleration. Similar to instantaneous angular velocity the instantaneous angular acceleration of the object at a particular instant is same as average angular acceleration of the object in a infinitesimally small duration at that particular instant.

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt} \quad \text{therefore}$$

$$\vec{\alpha} = \frac{d^2\vec{\theta}}{dt^2} = \frac{d\vec{\omega}}{dt} = \omega\frac{d\vec{\omega}}{d\theta}$$

Equations of motion in angular form :

(valid only for constant angular acceleration)

$$\omega_f = \omega_i + \alpha t$$

$$\theta = \omega_i t + \tfrac{1}{2}\alpha t^2$$

$$\omega_f^{\,2} = \omega_i^{\,2} + 2\alpha\theta$$

$$\langle\omega\rangle = \frac{\omega_i + \omega_f}{2}$$

$$\theta_{n^{\text{th}}} = \omega_i + \frac{\alpha}{2}(2n - 1)$$

Relation between tangential and angular velocity :

Since $\Delta\theta = \frac{\Delta s}{R}$ therefore

$$\Delta s = R\Delta\theta$$

$$\Rightarrow \quad \frac{\Delta s}{\Delta t} = R\frac{\Delta\theta}{\Delta t}$$

$$\Rightarrow \quad v = R\omega \qquad \text{.... (2)}$$

The above relation in vector form is

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} \quad \text{or} \quad \vec{\boldsymbol{\omega}} = \frac{\vec{\mathbf{r}} \times \vec{\mathbf{v}}}{r^2}$$

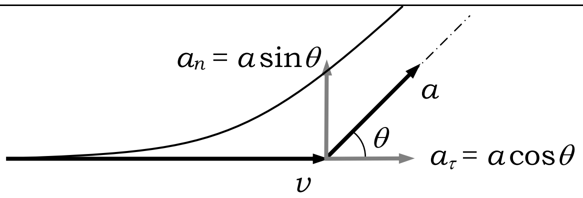
Here v is the tangential velocity of the particle and ω is its angular velocity. Differentiating (2) we get

$$\frac{dv}{dt} = R\frac{d\omega}{dt}$$

$$\Rightarrow \quad a = R\alpha \qquad \text{.... (3)}$$

Where a is tangential acceleration and α the angular acceleration of the particle.

When the direction of instantaneous acceleration of the particle and direction of its instantaneous velocity does not lie in same line at a particular instant, the particle moves on a curved path, which keeps bending to become asymptotic to the acceleration. The component of acceleration in the direction of tangential velocity at any instant (a_τ) is called its tangential acceleration while the component of acceleration perpendicular to its instantaneous velocity is called its normal acceleration (a_n). Tangential acceleration of the particle is responsible for change in magnitude of the velocity vector at any instant while the normal acceleration changes the direction of velocity vector only.



$$a_\tau = \frac{d}{dt}|\vec{\mathbf{v}}|$$

Let the instantaneous velocity and instantaneous acceleration be

$$\vec{\mathbf{v}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} \text{ and } \vec{\mathbf{a}} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}$$

$$\Rightarrow \quad v = \sqrt{{v_x}^2 + {v_y}^2}$$

the rate of change of speed is

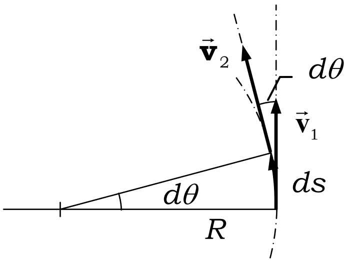
$$\Rightarrow \quad \frac{dv}{dt} = \frac{d}{dt}|\vec{\mathbf{v}}| = \frac{\left\{2v_x\frac{dv_x}{dt} + 2v_y\frac{dv_y}{dt}\right\}}{2\sqrt{{v_x}^2 + {v_y}^2}}$$

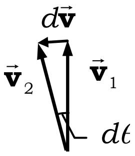
$$\Rightarrow \quad \frac{d}{dt}|\vec{\mathbf{v}}| = \frac{\{v_x a_x + v_y a_y\}}{\sqrt{{v_x}^2 + {v_y}^2}}$$

$$\Rightarrow \quad \frac{d}{dt}|\vec{\mathbf{v}}| = \frac{\vec{\mathbf{a}}.\vec{\mathbf{v}}}{v} = \frac{av\cos\theta}{v}$$

$$\Rightarrow \quad \frac{d}{dt}|\vec{\mathbf{v}}| = a\cos\theta = a_\tau$$

To find value of normal acceleration let us assume circular motion of a particle with uniform angular velocity (the speed of the particle being constant while moving on the circle) therefor no tangential acceleration. Thus the acceleration causing the change in velocity due to change in its direction can be calculated.





From figure for $\lim\limits_{d\theta \to 0}$

$$|\vec{\mathbf{v}}_1| = |\vec{\mathbf{v}}_2| = v \quad \text{therefore}$$

$$\frac{dv}{v} = d\theta = \frac{ds}{R}$$

$$\Rightarrow \quad dv = v\frac{ds}{R}$$

Thus the acceleration of the particle is

$$a_n = \frac{dv}{dt} = v\frac{ds}{Rdt} = \frac{v^2}{R}$$

Since the acceleration is ⊥ to the tangential velocity therefore it is called normal acceleration. It is also called centripetal acceleration because it is always directed towards the centre of the circle.

A more reasonable treatment of motion of object in circle is

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$$

$$\Rightarrow \quad \vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d\vec{\boldsymbol{\omega}}}{dt} \times \vec{\mathbf{r}} + \vec{\boldsymbol{\omega}} \times \frac{d\vec{\mathbf{r}}}{dt}$$

$$\Rightarrow \quad \vec{\mathbf{a}} = \vec{\boldsymbol{\alpha}} \times \vec{\mathbf{r}} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}} = \vec{\mathbf{a}}_\tau + \vec{\boldsymbol{\omega}} \times \left(\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}\right)$$

$$\Rightarrow \quad \vec{\mathbf{a}} = \vec{\mathbf{a}}_\tau - \omega^2 \vec{\mathbf{r}}$$

$$\Rightarrow \quad \vec{\mathbf{a}} = \vec{\mathbf{a}}_\tau + \vec{\mathbf{a}}_n \quad \text{where}$$

$$\vec{\mathbf{a}}_n = -\omega^2 \vec{\mathbf{r}}$$

here $\vec{\mathbf{a}}_n$ is directed towards the centre (opposite to $\vec{\mathbf{r}}$ vector) and $\vec{\mathbf{a}}_\tau$ is directed along the tangent to the circle (perpendicular to $\vec{\mathbf{r}}$ vector) are mutually perpendicular. Hence

$$a = \sqrt{{a_\tau}^2 + {a_n}^2}$$

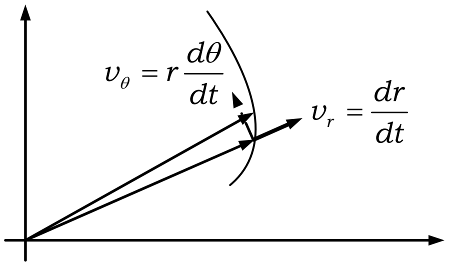
Velocity of particle in polar form :

In polar form two components of particles velocity are considered. One is radial component along the radius vector (the position vector) given by

$$v_r = \frac{dr}{dt}$$

And the other component perpendicular to radius vector is given by

$$v_\theta = r\frac{d\theta}{dt}$$



The resultant velocity of the particle is

$$\left|\vec{\mathbf{v}}\right| = \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2}$$

<u>Radius of curvature of trajectory</u> :

The small portion of a curve around a particular point can be assumed as the part of a circle. The radius of that circle is called radius of curvature of the curve at that particular point. To find radius of curvature of trajectory at a point, find the component of instantaneous acceleration of the particle perpendicular to velocity of the particle at that instant and put this equal to centripetal acceleration $\left({v^2}/{R}\right)$.

The radius of curvature of a trajectory $y = f(x)$ can also be found as

$$R = \left|\frac{\left\{1 + \left({dy}/{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2y}{dx^2}}\right|$$ [!]

[!] proof of this relation is beyond the scope of this text.

Representative example 4 :

A particle is projected with a velocity u at an angle α. Find the radius of curvature of trajectory of the particle t sec after the projection.

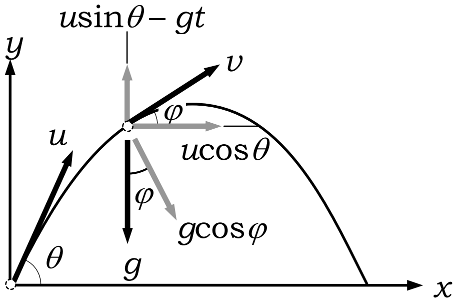
Solution :

Let the magnitude of velocity t sec after projection becomes v directed at an angle φ from the horizontal then the acceleration component perpendicular to instantaneous velocity is

$$g\cos\varphi$$

if the radius of curvature of the trajectory at this place is R than

$$g\cos\varphi = v^2/R$$



$$\Rightarrow \quad R = \frac{v^2}{g\cos\varphi} \quad \text{.... (1)}$$

in a projectile motion horizontal component of velocity remains same, hence

$$v\cos\varphi = u\cos\theta$$

$$\Rightarrow \quad \cos\varphi = \frac{u\cos\theta}{v} \quad \text{.... (2)}$$

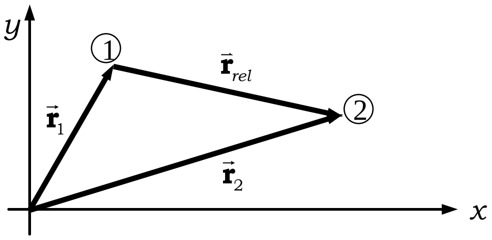
from (1) and (2)

$$R = \frac{v^3}{gu\cos\theta} = \frac{u^2\cos^2\theta}{g\cos^3\varphi}$$

$$\Rightarrow \quad R = \frac{\left(u^2 + g^2t^2 - 2ugt\sin\theta\right)^{3/2}}{gu\cos\theta}$$

RELATIVE VELOCITY :

In the adjoining diagram the position vector of point 1 is $\vec{\mathbf{r}}_1$ and of point 2 is $\vec{\mathbf{r}}_2$ relative to origin of co-ordinate of reference frame. Position vector of point 2 relative to point 1 is a vector from point 1 to point 2, which is



$$\vec{\mathbf{r}}_{rel} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$$

Differentiating w.r.t. time we get

$$\vec{\mathbf{v}}_{rel} = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1$$

Here $\vec{\mathbf{v}}_{rel}$ represents relative velocity of point 2 relative to point 1. Or in other wards the above relation can be written as

$$\vec{\mathbf{v}}_{rel} = \vec{\mathbf{v}}_{object} - \vec{\mathbf{v}}_{observr}\,. \qquad \text{.... (1)}$$

It is a vector relation and when relative velocity of object with respect to observer is calculated then the velocity of observer is taken zero and object's velocity as its relative velocity. Some times the relation is also used as

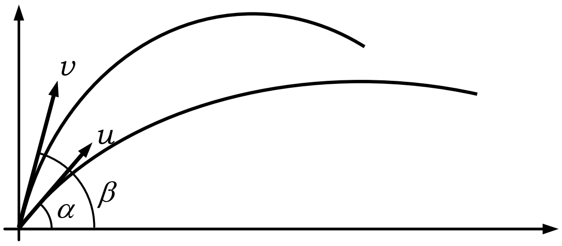
$$\vec{\mathbf{v}}_{object} = \vec{\mathbf{v}}_{rel} + \vec{\mathbf{v}}_{observr}\,. \qquad \text{.... (2)}$$

The actual velocity of an object (velocity of object in ground frame) is the vector sum of its relative velocity and the velocity of observer with respect to which it was calculated. Differentiating (1) gives relative acceleration

$$\vec{\mathbf{a}}_{rel} = \vec{\mathbf{a}}_{object} - \vec{\mathbf{a}}_{observr}\,. \qquad \text{.... (3)}$$

A projectile as seen from another projectile :

It is always a straight line. If vertical velocity components are equal one projectile moves relative to other on a horizontal straight line. Similarly if horizontal velocity components are equal one moves relative to the other on a vertical straight line.



$x_1 = u\cos\alpha t$, $x_2 = v\cos\beta t$ similarly

$y_1 = u\sin\alpha t - \tfrac{1}{2}gt^2$, $y_2 = v\sin\beta t - \tfrac{1}{2}gt^2$

displacement of 2 relative to 1 is

$$X = x_2 - x_1 = (v\cos\beta - u\cos\alpha)t$$

$$Y = y_2 - y_1 = (v\sin\beta - u\sin\alpha)t$$

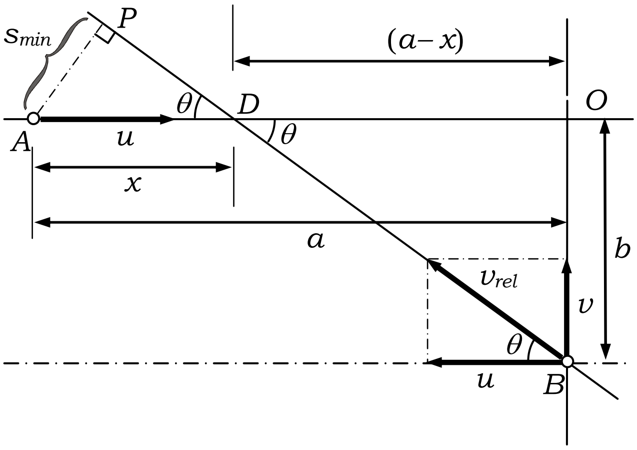
$$\Rightarrow \quad Y = \left(\frac{v\sin\beta - u\sin\alpha}{v\cos\beta - u\cos\alpha}\right)X$$

is a straight line passing through origin.

Representative example 5 :

Two cars A and B moving on two intersecting mutually perpendicular roads have uniform velocity u and v respectively. Their initial distances from the point of intersection are a and b respectively. Calculate the minimum distance of approach between the two cars.

Solution :



v_{rel} represents the relative velocity of car B relative to car A. Now car A can be assumed stationary and B moving on the line BD. Therefore the minimum distance of approach between the cars is

$$s_{min} = x\sin\theta$$

From triangle OBD

$$\tan\theta = \frac{b}{a-x} = \frac{v}{u}$$

$$\Rightarrow \quad x = a - \frac{b}{\left(v/u\right)}$$

$$\Rightarrow \quad x = \frac{av - bu}{v}$$

since $\tan\theta = v/u$ therefore

$$\sin\theta = \frac{v}{\sqrt{v^2+u^2}} \quad \text{and therefore}$$

$$s_{min} = x\sin\theta$$

$$\Rightarrow \quad s_{min} = \frac{av-bu}{v}\,\frac{v}{\sqrt{v^2+u^2}}$$

$$\Rightarrow \quad s_{min} = \frac{av-bu}{\sqrt{v^2+u^2}}$$

time taken to reach the minimum distance of approach

$$\tau = \frac{BD + DP}{v_{rel}}$$

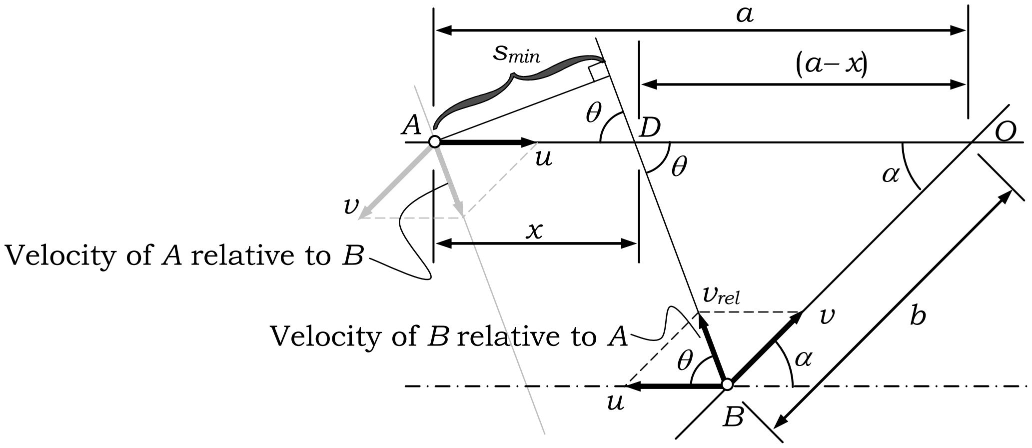
$$\Rightarrow \quad \tau = \frac{\frac{b}{\sin\theta} + x\cos\theta}{v_{rel}}$$

$$\Rightarrow \quad \tau = \frac{\frac{b\sqrt{v^2+u^2}}{v} + \left(\frac{av-bu}{v}\right)\frac{u}{\sqrt{v^2+u^2}}}{\sqrt{v^2+u^2}}$$

$$\Rightarrow \quad \tau = \frac{(au+bv)}{v^2+u^2}$$

Representative example 6 :

Two cars A and B moving on two intersecting roads at an angle α have uniform velocity u and v respectively. Their initial distances from the point of intersection are a and b respectively. Calculate the minimum distance of approach between the two cars.



Solution :

v_{rel} represents the relative velocity of car B relative to car A. Now car A can be assumed stationary and B moving on the line BD. Therefore the minimum distance of approach between the cars is

$$s_{min} = x\sin\theta$$

From triangle OBD

$$\frac{b}{\sin\theta} = \frac{a-x}{\sin(\pi-(\alpha+\theta))} = \frac{a-x}{\sin(\alpha+\theta)}$$

$$\Rightarrow \quad x = a - b\frac{\sin(\alpha+\theta)}{\sin\theta}$$

$$\Rightarrow \quad x = a - b\left(\frac{\sin\alpha}{\tan\theta}+\cos\alpha\right)$$

but from figure

$$\tan\theta = \frac{v\sin(\pi-\alpha)}{u+v\cos(\pi-\alpha)}$$

$$\Rightarrow \quad \tan\theta = \frac{v\sin\alpha}{u-v\cos\alpha}$$

$$\Rightarrow \quad \left(\frac{\sin\alpha}{\tan\theta}+\cos\alpha\right) = \frac{u}{v} \quad \text{and}$$

$$\sin\theta = \frac{v\sin\alpha}{\sqrt{u^2+v^2-2uv\cos\alpha}}$$

Therefore

$$x = a - b\,\frac{u}{v} = \frac{av-bu}{v}$$

$$\Rightarrow \quad s_{min} = x\sin\theta$$

$$\Rightarrow \quad s_{min} = \frac{av-bu}{v}\,\frac{v\sin\alpha}{\sqrt{u^2+v^2-2uv\cos\alpha}}$$

$$\Rightarrow \quad s_{min} = \frac{(av-bu)\sin\alpha}{\sqrt{u^2+v^2-2uv\cos\alpha}}$$

The velocity of A relative to B $(\vec{\mathbf{u}} - \vec{\mathbf{v}})$ and the velocity of B relative to A $(\vec{\mathbf{v}} - \vec{\mathbf{u}})$ are anti-parallel vectors and the minimum distance of approach is the perpendicular distance between the lines of motion of A relative to B and B relative to A.

time taken to reach the minimum distance of approach

$$\tau = \frac{BD + DP}{v_{rel}}$$

$$\Rightarrow \quad \tau = \frac{b\dfrac{\sin\alpha}{\sin\theta} + x\cos\theta}{v_{rel}}$$

$$\Rightarrow \quad \tau = \frac{b\dfrac{\sqrt{u^2 + v^2 - 2uv\cos\alpha}}{v} + \left(\dfrac{av - bu}{v}\right)\dfrac{u - v\cos\alpha}{\sqrt{u^2 + v^2 - 2uv\cos\alpha}}}{\sqrt{u^2 + v^2 - 2uv\cos\alpha}}$$

$$\Rightarrow \quad \tau = \frac{b\left(u^2 + v^2 - 2uv\cos\alpha\right) + (av - bu)(u - v\cos\alpha)}{\left(u^2 + v^2 - 2uv\cos\alpha\right)v}$$

$$\Rightarrow \quad \tau = \frac{(au + bv) - (bu + av)\cos\alpha}{\left(u^2 + v^2 - 2uv\cos\alpha\right)}$$

For the two particles moving on intersecting lines to collide at the intersection, the minimum distance of approach between them should be zero. Hence

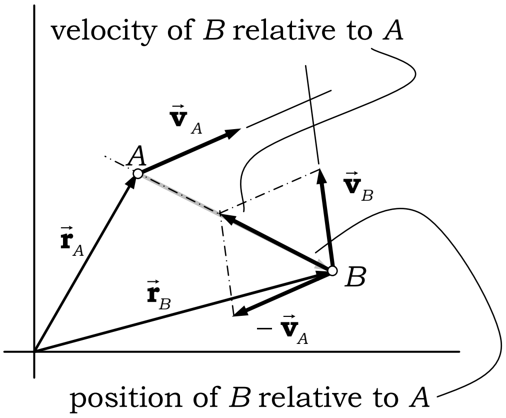
$$\frac{(av - bu)\sin\alpha}{\sqrt{u^2 + v^2 - 2uv\cos\alpha}} = 0$$

$$\Rightarrow \quad av - bu = 0$$

$$\Rightarrow \quad a/u = b/v$$

means both the particles reach intersection simultaneously, or the time A takes to reach the intersection $\left(a/u\right)$ is same as the time B takes to reach the intersection $\left(b/v\right)$.

If A and B are going to collide at the intersection then the direction of motion of B relative to A must pass through instantaneous position of A. Then only, the minimum distance of approach between A and B will become zero. In other words, for the collision of two particles moving on intersecting straight lines the velocity of B relative to A is opposite to the position vector of B relative to A (and visa versa).



$$(\vec{\mathbf{v}}_B - \vec{\mathbf{v}}_A) \uparrow\downarrow (\vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A)$$

mathematically it means that the two unit vectors, along relative velocity and along relative position are related as

$$\frac{(\vec{\mathbf{v}}_B - \vec{\mathbf{v}}_A)}{|\vec{\mathbf{v}}_B - \vec{\mathbf{v}}_A|} = -\frac{(\vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A)}{|\vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A|}$$

$$\Rightarrow \quad \frac{(\vec{\mathbf{v}}_B - \vec{\mathbf{v}}_A)}{|\vec{\mathbf{v}}_B - \vec{\mathbf{v}}_A|} = \frac{(\vec{\mathbf{r}}_A - \vec{\mathbf{r}}_B)}{|\vec{\mathbf{r}}_A - \vec{\mathbf{r}}_B|}$$

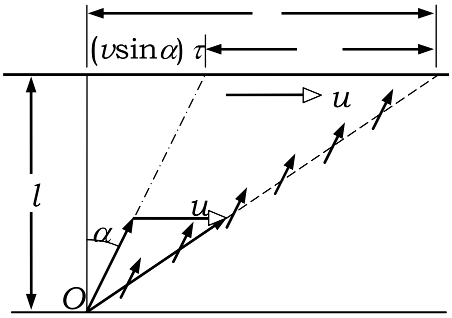
River–Swimmer problem :

Let u = velocity of water,

v = velocity of swimmer relative to water, it is also expressed as velocity of swimmer in still water.

α = is the angle between the swimmer's velocity relative to water and perpendicular to the flow of water.

Time taken to cross the river is always calculated by component of swimmer's velocity perpendicular to river flow



$$\tau = \frac{l}{v\cos\alpha}$$

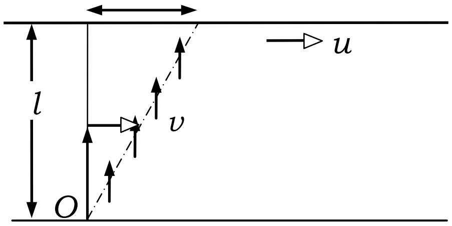
Drift distance

$$s = (v\sin\alpha + u)\tau$$

$$\Rightarrow \quad s = (v\sin\alpha + u)\frac{l}{v\cos\alpha}$$

⌖ crossing the river in minimum time :

To cross the river in minimum time (τ_{min}), $\cos\alpha$ should be maximum, thus



$$\cos\alpha = 1 = \cos(0)$$

$$\Rightarrow \quad \alpha = 0.$$

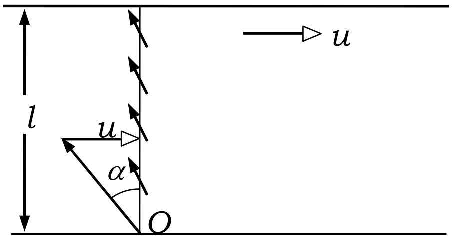
$\Rightarrow \quad \tau_{min} = \dfrac{l}{v}$

Therefore the swimmer will be able to cross the river in minimum time if he swims perpendicular to river flow. In this case he will not be reaching the opposite point on the other bank but he will be drifted by

$$s = u\tau_{min} = \frac{ul}{v}$$

⌖ **crossing the river in minimum distance** :

case 1: (magnitude of swimmer's velocity > river flow velocity)



To cross the river in minimum distance, drift distance s should be zero, for this swimmer should swim at some angle perpendicular to river flow opposite to flow as shown such that

$$v \sin\alpha - u = 0$$

Therefore the swimmer will able to cross the river in minimum distance, that is with zero drift when its resultant velocity is
along the perpendicular to flow. Thus to achieve zero drift

$$\sin\alpha = {}^{u}\!/_{v}$$

but this is possible only when swimmer can swim with a velocity greater in magnitude compared to river flow velocity. In this case time of journey is

$$\tau = \frac{d}{v\cos\alpha}$$

$$\Rightarrow \quad \tau = \frac{d}{\sqrt{v^2 - u^2}}$$

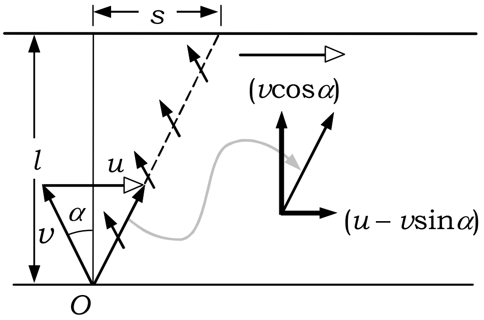
and drift distance

$$s = 0$$

case 2: (magnitude of swimmer's velocity < river flow velocity)

From above condition it is clear that zero drift can be achieved only when ($v > u$), otherwise $\sin\alpha$ will be greater than 1 which will not allow α to take any meaning full value. In that case drift can not be made zero, but still α can be chosen in such a way that drift becomes minimum. Therefore for minimum drift when ($v < u$),

$$\frac{ds}{d\alpha} = \frac{d}{d\alpha}\left\{\frac{l(u - v\sin\alpha)}{v\cos\alpha}\right\}$$



$$\Rightarrow \quad \frac{ds}{d\alpha} = \frac{l}{v}\left(u\sec\alpha\tan\alpha - v\sec^2\alpha\right) = 0$$

$$\Rightarrow \quad u\sin\alpha - v = 0$$

$$\Rightarrow \quad \sin\alpha = {}^{v}\!/_{u}$$

If the swimmer swims at this angle he reaches the opposite bank of the river with minimum drift (s_{min}). In this case time of journey is

$$\tau = \frac{d}{v\cos\alpha}$$

$$\Rightarrow \quad \tau = \frac{ud}{v\sqrt{u^2 - v^2}}$$

and the minimum drift distance

$$s_{min} = (u - v\sin\alpha)\tau$$

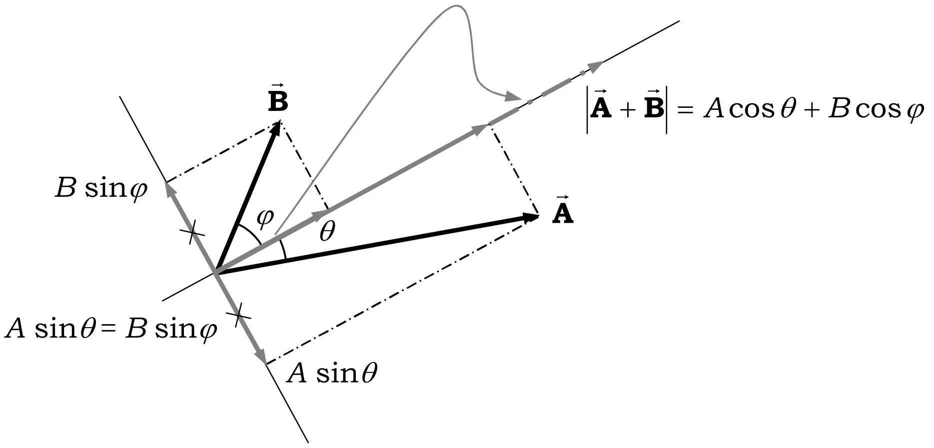
$$\Rightarrow \quad s_{min} = \frac{(u^2 - v^2)}{u}\frac{ud}{v\sqrt{u^2 - v^2}}$$

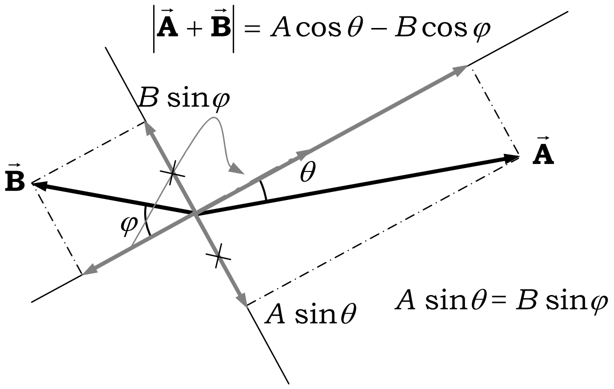
$$\Rightarrow \quad s_{min} = \frac{d\sqrt{u^2 - v^2}}{v}$$

Resultant velocity :

Velocity of an object in ground frame is the vector sum of its relative velocity and the actual velocity of observer. Some times it is desired to have the velocity of an object in a particular direction. For example an aeroplane has to move from point *A* to *B* in a blowing wind. To be able to move in a certain direction in ground frame the actual velocity of the plane (vector sum of its velocity relative to wind and the velocity of wind) must be directed from *A* to *B*.

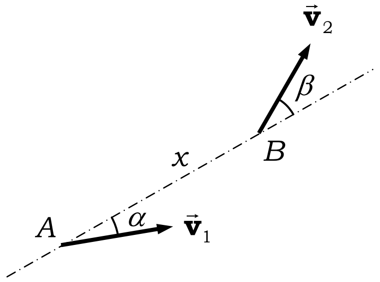
When we want the resultant of two vectors to fall in a certain direction, it is possible only when the components of two vectors perpendicular to that certain direction exactly cancel each other. In this case the resultant of these vectors along that certain direction is the sum (or difference depending on the direction of components) of components of two vectors in that direction.

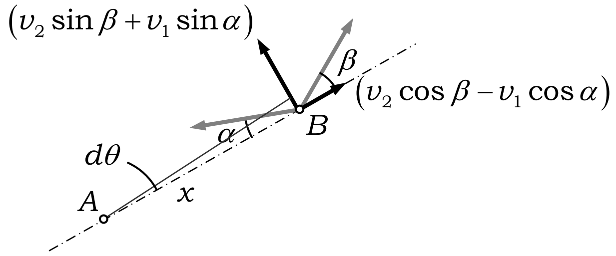




Relative angular velocity and angular acceleration :

If two particles are moving with velocities $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ in a plane and at a certain instant distance between them is x, then





Angular velocity of B relative to A is defined as

$$\omega_{rel} = \frac{v_2\sin\beta + v_1\sin\alpha}{x} = \frac{d\theta}{dt}$$

From the diagram

$$\frac{dx}{dt} = \left(v_2\cos\beta - v_1\cos\alpha\right)$$

Therefore angular acceleration of B relative to A

$$\alpha_{rel} = \frac{d\omega_{rel}}{dt}$$

$$\Rightarrow \quad \alpha_{rel} = -\frac{\left(v_2\sin\beta + v_1\sin\alpha\right)}{x^2}\frac{dx}{dt}$$

$$\Rightarrow \quad \alpha_{rel} = -\frac{\left(v_2\sin\beta + v_1\sin\alpha\right)\left(v_2\cos\beta - v_1\cos\alpha\right)}{x^2}$$

Important to note :
For two mutually interacting particles approaching obliquely, like two celestial objects (considered to be particles) experiencing mutual gravitational attraction or two point charges (like or unlike) experiencing Coulombic interaction, at the minimum distance of approach the instantaneous position of one relative to the other is perpendicular to the relative velocity of one relative to the other. It is because the at minimum distance of approach they must not have the component of relative velocity in the line of their relative position. (For r_{rel} to be minimum $\frac{dr_{rel}}{dt} = 0 = \upsilon_{rel}$)