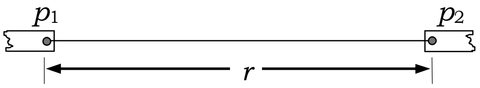
MAGNETISM

MAGNETOSTATICS

Force between two magnetic monopoles :
Although the magnetic monopole is a hypothetical concept, (it does not exists because of one of the important property of magnetic lines of force that these form a closed loop) but still it works on the two poles of a dipole separately. The magnitude of magnetic force between two magnetic poles is directly proportional to the amount of pole-strengths (p) and inversely proportional to the square of the distance between the two (analogous to point charge).



$$F \propto p_1,\ p_2 \quad \text{and}$$

$$F \propto \frac{1}{r^2}$$

$$\Rightarrow \qquad F \propto \frac{p_1 p_2}{r^2}$$

$$\Rightarrow \qquad F = K' \frac{p_1 p_2}{r^2}$$

If between the two poles there is free space then

$$K' = \frac{\mu_o}{4\pi} = 10^{-7}\ \text{WbA}^{-1}\text{m}^{-1},$$

where μ_o is the magnetic permeability of the free space.

$$\mu_o = 1.256 \times 10^{-6}\ \text{WbA}^{-1}\text{m}^{-1}.$$

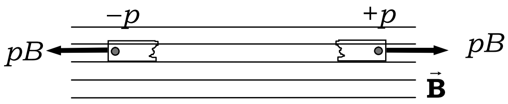
Magnetic poles are of two types. Positive pole represents north-pole where as negative for south-pole. As in case of charges, like magnetic poles repel each other where as the unlike poles attract each other.

Magnetic Field :
The region of influence of a magnetic pole (or current carrying element) in which some other magnetic pole experiences the Magnetic force is called the magnetic field of that magnetic pole (or current carrying element).

Magnetic field consists of magnetic lines of force. Magnetic lines of force have following properties.

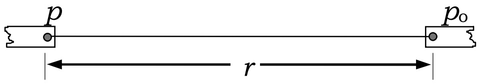
i) They do not cross each other.
ii) They form closed loops. Traveling form +ve pole (North pole) to –ve pole (South pole), outside the magnet and form –ve pole (South pole) to +ve pole (North pole), inside the magnet. They encircle the current carrying element.

Force on a magnetic pole inside a magnetic field : Force on a magnetic pole inside a magnetic field of strength B is the product of pole strength p and magnetic field. +ve pole (North pole) experiences this force in the direction of field where as –ve pole (South pole) experiences force opposite to the field. Hence force on a magnetic monopole placed inside a magnetic field is



$\vec{\mathbf{F}} = p\vec{\mathbf{B}}$.

Magnetic Field Strength :
Magnetic field strength at a point inside the field is defined as the force per unit positive pole-strength. Magnetic field strength due to a magnetic monopole is



$$B = \frac{F}{p_o} = \frac{\mu_o}{4\pi}\frac{p}{r^2}$$

$$\vec{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{p}{r^3}\vec{\mathbf{r}}$$

The magnetic field follows the law of superposition. Suppose $\vec{\mathbf{B}}_1$ is the field at a point due to a pole p_1 and $\vec{\mathbf{B}}_2$ is the field at the same point due to a pole p_2. The resultant field when both the poles are present, is

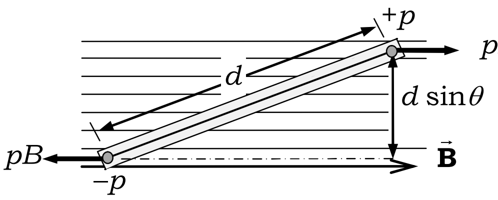
$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 .$$

Positive pole (North pole) generates magnetic field away from it where as magnetic field due to negative pole (south pole) is towards it.
Magnetic potential and magnetic potential energy is out of the scope of our discussion.

Magnetic dipole :
A bar magnet consisting of two magnetic poles, whose strengths are always equal in magnitude and opposite in nature, separated by a certain distance d (for the convenience of calculation people take this distance to be $2l$) is called the magnetic dipole.

Magnetic dipole moment :
Is defined as a vector $\vec{\mathbf{M}} = p\left(\vec{\mathbf{d}}\right)$, where the $\vec{\mathbf{d}}$ is the vector starting from south pole (–ve) and terminating at north pole (+ve) in the magnetic dipole.

magnetic diapole placed in a uniform magneic field :
Torque experienced by an magnetic dipole placed in a uniform magnetic field :



$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{d}} \times p\vec{\mathbf{B}} = \vec{\mathbf{M}} \times \vec{\mathbf{B}}$$

$\Rightarrow \quad \tau = pB\,d\sin\theta$

$\Rightarrow \quad \tau = MB\sin\theta,$

Work done in rotating the magnetic dipole placed in a uniform magnetic field :

$$W = \int_0^\theta \tau \, d\theta$$

$$\Rightarrow \quad W = \int_0^\theta pB\, d \sin\theta \, d\theta$$

$$\Rightarrow \quad W = MB\left(1 - \cos\theta\right)$$

Oscillation time period of the magnetic dipole placed in a uniform electric field for small oscillations:
Restoring torque on dipole at small angular displacement θ

$$\tau = -pB\, d \sin\theta = I\alpha$$

$\Rightarrow \quad -MB\,\theta = I\,\alpha$ (as θ is small, $\sin\theta = \theta$)

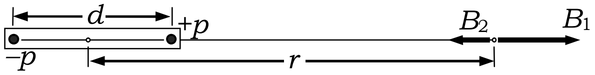
where I is the moment of inertia of the dipole about an axis passing through its midpoint and α is its angular acceleration. Therefore

$$T = 2\pi\sqrt{-\frac{\text{displacement}}{\text{acceleration}}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{-\frac{\theta}{\alpha}} = 2\pi\sqrt{\frac{I}{MB}}$$

Magnetic field due to magnetic dipole :

Axial position :
If B_1 is the field produced by $+p$ (north pole) and B_2 because of $-p$ (south pole) then



$$B_1 = \frac{\mu_o}{4\pi}\frac{p}{\left(r - d/2\right)^2} \text{ and}$$

$$B_2 = \frac{\mu_o}{4\pi}\frac{p}{\left(r + d/2\right)^2}$$

therefore the net field produced is

$$B = B_1 - B_2$$

$$\Rightarrow \quad B = \frac{\mu_o p}{4\pi}\left\{\frac{1}{\left(r - d/2\right)^2} - \frac{1}{\left(r + d/2\right)^2}\right\}$$

$$\Rightarrow \quad B = \frac{\mu_o}{4\pi}\frac{p \times 2rd}{\left(r^2 - d^2/4\right)^2}$$

$$\Rightarrow \quad B = \frac{\mu_o}{4\pi}\frac{2Mr}{\left(r^2 - d^2/4\right)^2}$$

If $r >> d$ then

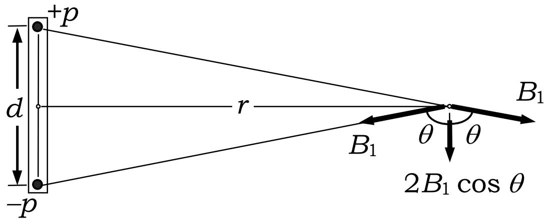
$$B \approx \frac{\mu_o}{4\pi}\frac{2M}{r^3}$$

The above formulae for magnetic field at the axis of dipole are valid only for points lying out side the dipole on both sides of the dipole.

It is important to note that the direction of net magnetic field is same as the direction of magnetic dipole moment in this case.

Equatorial position :

Since the point of interest is equidistant from both +*p* and –*p* therefore the magnitude of the field produced by both the poles will be equal. If B_1 is the field produced by +*p* then



$$B_1 = \frac{\mu_o}{4\pi}\frac{p}{\left(r^2 + \frac{d^2}{4}\right)} \text{ thus}$$

$$B = 2B_1 \cos\theta$$

$$\Rightarrow \quad B = \frac{2\mu_o}{4\pi}\frac{p}{\left(r^2 + \frac{d^2}{4}\right)}\frac{\frac{d}{2}}{\sqrt{\left(r^2 + \frac{d^2}{4}\right)}}$$

$$\Rightarrow \quad B = \frac{\mu_o}{4\pi}\frac{M}{\left(r^2 + \frac{d^2}{4}\right)^{3/2}}$$

If $r >> d$ then

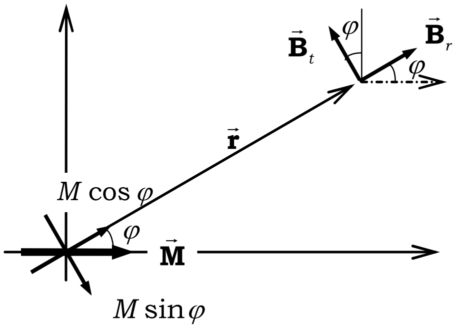
$$B \approx \frac{\mu_o}{4\pi}\frac{M}{r^3}.$$

Note that in this position the direction of resultant magnetic field is opposite to the direction of magnetic dipole moment vector.

One another thing important to note is that this analogues treatment to the magnetic dipole is not valid for the points between the poles of the dipole, as the direction of magnetic field due to monopoles changes between the poles.

General position (for small dipole) :

Taking component of the dipole moment along and ⊥ directions of the radius vector to point of interest we get the point of interest at axial position for the component along the radius vector and it is at equatorial position for the component ⊥ to the radius vector.



The radial component of magnetic field

$$B_r = \frac{\mu_o}{4\pi}\frac{2M\cos\varphi}{r^3}$$

and the transverse component of field

$$B_t = \frac{\mu_o}{4\pi}\frac{(-M\sin\varphi)}{r^3}$$

$$B_{\text{net}} = \sqrt{B_r^2 + B_t^2}$$

$$\Rightarrow \quad B_{\text{net}} = \frac{\mu_o}{4\pi}\frac{M}{r^3}\sqrt{1+3\cos^2\varphi}$$

The value of angle φ for which resultant magnetic field is perpendicular to dipole moment vector.

$$B_r\cos\varphi = B_t\sin\varphi$$

$$\Rightarrow \quad \frac{\mu_o}{4\pi}\frac{2M\cos\varphi}{r^3}\cos\varphi = \frac{\mu_o}{4\pi}\frac{M\sin\varphi}{r^3}\sin\varphi$$

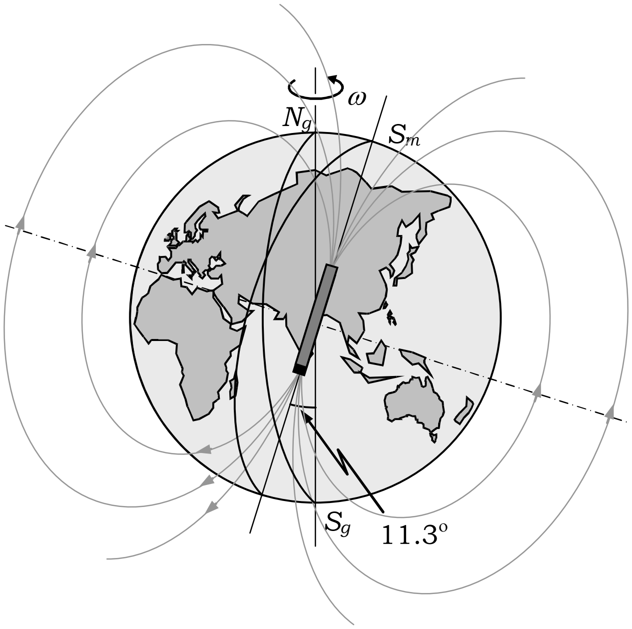
$$\Rightarrow \quad 2\cos^2\varphi = \sin^2\varphi$$

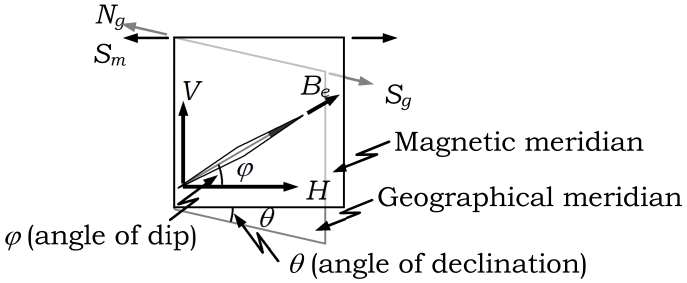
$$\Rightarrow \quad \varphi = \tan^{-1}\left(\sqrt{2}\right)$$

EARTH'S MAGNETISM

Magnetic and Geographical meridian :

The vertical plane at the position of observer indicating the direction of magnetic North and magnetic South Pole is called magnetic meridian. Similarly the vertical plane at the position of observer indicating the direction of geographical north and geographical south is called geographical meridian.





Angle of declination (θ) :
The angle between magnetic and geographical meridian at the position of observer is called angle of declination. Angle of declination changes with the position on the globe.

$$0 \le \theta \le \pi$$

Angle of dip (φ) :
When a magnetic needle, free to rotate in vertical plane is placed in magnetic meridian (called dip needle) it aligns itself in the direction of resultant earth's magnetic field. The angle made by dip needle with horizontal in magnetic meridian is called dip angle (or angle of inclination).

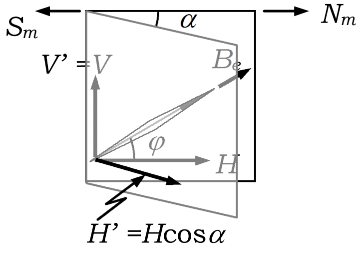
$$-\pi/2 \le \varphi \le \pi/2$$

H and V are the horizontal and vertical components of earths magnetic field at the position of observer, then

$$H = B_e \cos\varphi \text{ and } V = B_e \sin\varphi \text{ hence}$$

$$\tan\varphi = V/H \text{ and } B_e = \sqrt{V^2 + H^2}$$

Calculation of apparent dip angle in a meridian other than magnetic meridian :
In another vertical plane (other than magnetic meridian) the vertical component of earth's magnetic field remains same but horizontal component of earth's magnetic field is the component of H in magnetic meridian, thus



$$V' = V \text{ and } H' = H\cos\alpha$$

if apparent dip angle is φ' then

$$\tan\varphi' = V'/H' = \frac{V}{H\cos\alpha}$$

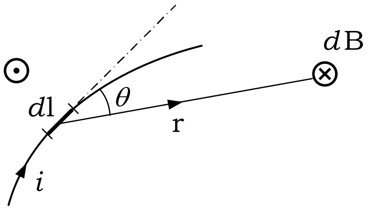
$$\Rightarrow \quad \tan\varphi' = \frac{\tan\varphi}{\cos\alpha}$$

It is clear that as α increases, φ' also increases and becomes $\pi/2$ when α becomes $\pi/2$, because H' becomes zero.

MAGNETIC EFFECT OF CURRENT

Oersted's experiment :
In 1820 Danish physicist Hans Christian Oersted observed that current carrying wire can deflect a magnetic needle, hence it actually produces a magnetic field. This gave way to unification of two branches of physics namely Electricity and Magnetism (the Electromagnetism).

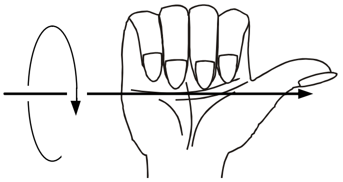
BIOT-SAVART's LAW :
The magnetic field at a point P due to a current element is given by



$$d\vec{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{\left(id\vec{\mathbf{l}}\right)\times\vec{\mathbf{r}}}{r^3}$$

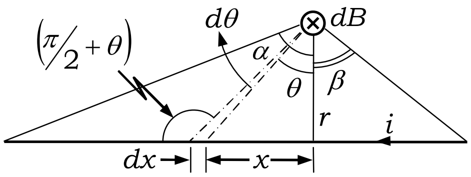
$$\Rightarrow \qquad dB = \frac{\mu_o}{4\pi}\frac{i\,d\ell\sin\theta}{r^2}$$

Here it is important to note that the direction of magnetic field is perpendicular to the plane containing the current carrying element and the line joining the element and the point of interest.



Maxwell's right handed cork screw law :
To find the direction of the magnetic lines of force close your right hand shooting your thumb out in the direction of current then the direction of the bend of the other fingers will give the magnetic field.

Magnetic field near a straight current carrying conductor :
The magnetic due to element dx is



$$dB = \frac{\mu_o}{4\pi}\frac{i\,dx\sin\left(\pi/2+\theta\right)}{\left(r/\cos\theta\right)^2}$$

$\Rightarrow \quad dB = \dfrac{\mu_0}{4\pi}\dfrac{i\,dx\cos^3\theta}{r^2}$,

From figure

$x = r\tan\theta$

$\Rightarrow \quad \dfrac{dx\cos^2\theta}{r} = d\theta$, therefore

$$B = \int_{-\alpha}^{\beta} \frac{\mu_0}{4\pi}\frac{i}{r}\cos\theta d\theta$$

$$\Rightarrow \quad B = \frac{\mu_0}{4\pi}\frac{i}{r}\left(\sin\theta\right)_{-\alpha}^{\beta}$$

$$\Rightarrow \quad B = \frac{\mu_0}{4\pi}\frac{i}{r}\left(\sin\alpha + \sin\beta\right)$$

cor 1 : For ∞ conductor $\alpha = \beta = \pi/2$. Therefore

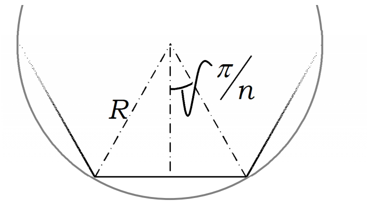
$$B = \frac{\mu_0 i}{2\pi r}$$

cor 2 : For semi-infinite rod $\alpha = 0$ and $\beta = \pi/2$. Therefore

$$B = \frac{\mu_0 i}{4\pi r}$$

Magnetic field at the centre of a regular polygon :

Magnetic field due to a current carrying wire, bent in the form of a regular polygon of n-sides with radius of circumscribing circle R. One side of the polygon subtends an angle $\dfrac{2\pi}{n}$ at the centre. Magnetic field due to one side of the polygon at the centre is



$$B_1 = \frac{\mu_0}{4\pi}\frac{i}{R\cos\left(\pi/n\right)}2\sin\left(\pi/n\right)$$

$$\Rightarrow \quad B_1 = \frac{\mu_0}{2\pi}\frac{i}{R}\tan\left(\pi/n\right)$$

hence magnetic field due to entire polygon

$$B = n\frac{\mu_0}{2\pi}\frac{i}{R}\tan\left(\pi/n\right)$$

$$\Rightarrow \quad B = \frac{\mu_0 i}{2R}\frac{\tan\left(\pi/n\right)}{\left(\pi/n\right)}$$

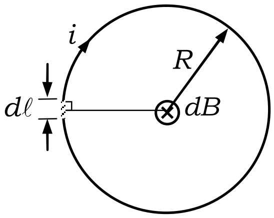
cor : When $n \to \infty$, polygon converts in to smooth circle hence magnetic field at the centre of a circular loop

$$B = \lim_{\pi/n \to 0} \frac{\mu_0 i}{2R} \frac{\tan\left(\pi/n\right)}{\left(\pi/n\right)}$$

$\Rightarrow$ $$B = \frac{\mu_0 i}{2R}$$

Magnetic field at the centre of a circular loop :
Let the loop of radius R consists of n turns. Magnetic field due to element dl is

$$dB = \frac{\mu_0}{4\pi} \frac{idl \sin\left(\pi/2\right)}{R^2}$$



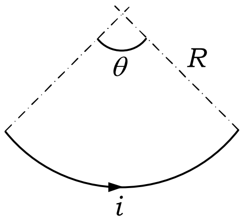
$\Rightarrow$ $$B = \oint \frac{\mu_0}{4\pi} \frac{id\ell}{R^2} = \frac{\mu_0}{4\pi} \frac{i}{R^2} \oint d\ell$$ Thus

$$B = \frac{\mu_0}{4\pi} \frac{i}{R^2} \times 2\pi R = \frac{\mu_0 i}{2R}.$$

For n turns

$$B = \frac{\mu_0 n i}{2R}$$

Magnetic field at the centre of curvature of an arc :
Since all elements of the arc contribute magnetic field in the same direction hence applying unitary method



$$B = \frac{\mu_0 i}{2R} \times \frac{\theta}{2\pi}$$

$\Rightarrow$ $$B = \frac{\mu_0}{4\pi} \frac{i\theta}{R}$$

Magnetic field at the center of a uniform circular loop is zero if a source of emf is connected between any point of it.
Resistance of lower part of circular wire with resistance per unit length λ is

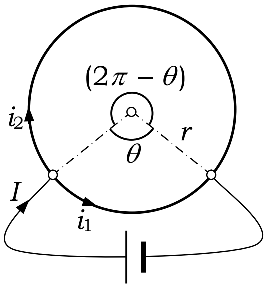
$$R_1 = r\theta\lambda$$

and resistance of the upper part is

$$R_2 = r(2\pi - \theta)\lambda$$

hence current in the lower part is

$$i_1 = \frac{R_2}{R_1 + R_2} I = \frac{(2\pi - \theta)}{2\pi} I$$



and current in the upper part is

$$i_2 = \frac{R_1}{R_1 + R_2} I = \frac{\theta}{2\pi} I$$

therefore magnetic field at the center of curvature due to lower part of the circular wire coming out of the plane of paper is

$$B_1 = \frac{\mu_o i_1}{2r}\left(\frac{\theta}{2\pi}\right) = \frac{\mu_o (2\pi - \theta) I}{4\pi r}\left(\frac{\theta}{2\pi}\right)$$

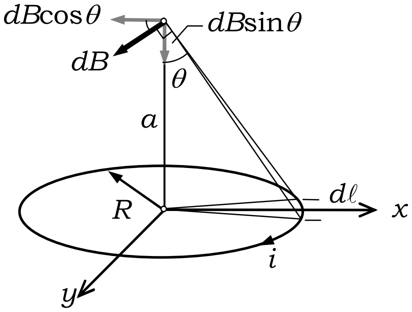
and the magnetic field due to upper part entering in to the plane of the paper is

$$B_2 = \frac{\mu_o i_2}{2r}\left(\frac{2\pi - \theta}{2\pi}\right) = \frac{\mu_o \theta I}{4\pi r}\left(\frac{(2\pi - \theta)}{2\pi}\right)$$

Since B_1 and B_2 are equal in magnitude and opposite in direction hence resultant magnetic field at the center of curvature is zero.

Magnetic field at the axis of a circular loop :

Magnetic field due to element $d\ell$ is (note : the element $d\ell$ is in the x–y plane where as the line joining the element and the point of interest lies in a plane perpendicular to x–y plane).



$$dB = \frac{\mu_o}{4\pi}\frac{i d\ell \sin\left(\pi/2\right)}{\left(R^2 + a^2\right)}$$

$$\Rightarrow \qquad dB = \frac{\mu_o}{4\pi}\frac{i d\ell}{\left(R^2 + a^2\right)}$$

As $B\cos\theta$ component gets canceled, the resultant field is

$$B = \oint dB \sin\theta$$

$$\Rightarrow \qquad B = \frac{\mu_o}{4\pi}\frac{i}{\left(R^2+a^2\right)}\frac{R}{\sqrt{R^2+a^2}}\oint d\ell$$

$$\Rightarrow \qquad B = \frac{\mu_o}{4\pi}\frac{iR}{\left(R^2+a^2\right)^{3/2}}\times 2\pi R$$

$$\Rightarrow \qquad B = \frac{\mu_o iR^2}{2\left(R^2+a^2\right)^{3/2}}$$

cor : Magnetic field at the centre of the loop can be found by putting $a = 0$, thus

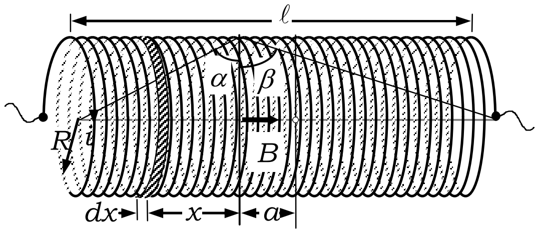
$$B = \mu_o\frac{iR^2}{2R^3} = \frac{\mu_o i}{2R}$$

If there are n turns in the loop then

$$B = \frac{\mu_o niR^2}{2\left(R^2+a^2\right)^{3/2}} \text{ and at the centre } B = \frac{\mu_o ni}{2R}$$

Magnetic field inside a solenoid :

Let a solenoid of length ℓ and total number of turns N caries a current i. Magnetic field at a distance a from the centre of the solenoid due to an element of thickness dx is



$$dB = \mu_o\left(\frac{N}{\ell}dx\right)i\frac{R^2}{2\left(R^2+x^2\right)^{3/2}}$$

$$\Rightarrow \qquad B = \mu_o\frac{NiR^2}{2\ell}\int_{-\left(\ell/2-a\right)}^{\left(\ell/2+a\right)}\frac{dx}{\left(R^2+x^2\right)^{3/2}},$$

substituting $x = R\tan\theta$

$$\Rightarrow \qquad dx = R\sec^2\theta\, d\theta,$$

Let $\qquad I = \int\frac{dx}{\left(R^2+x^2\right)^{3/2}}$

$$\Rightarrow \qquad I = \frac{1}{R^2}\int\cos\theta\, d\theta$$

$$\Rightarrow \qquad I = \frac{1}{R^2}\sin\theta$$

$$\Rightarrow \qquad I = \frac{1}{R^2}\left(\frac{x}{\sqrt{R^2+x^2}}\right), \text{ therefore}$$

$$B = \mu_o\frac{Ni}{2\ell}\left(\frac{x}{\sqrt{R^2+x^2}}\right)_{-\left(\ell/2-a\right)}^{\left(\ell/2+a\right)}$$

This expression in terms of angles subtended by ends as shown in the figure is

$$B = \mu_o \frac{Ni}{2\ell}(\sin\alpha + \sin\beta)$$

cor 1 : To calculate the magnetic field at the centre of the solenoid put a = 0, therefore

$$B = \mu_o \frac{Ni}{2\ell}\left(\frac{x}{\sqrt{R^2 + x^2}}\right)_{-\ell/2}^{\ell/2}$$

$$\Rightarrow \quad B = \mu_o \frac{Ni}{2\ell} 2 \frac{\ell}{\sqrt{4R^2 + \ell^2}}$$

$$\Rightarrow \quad B = \frac{\mu_o iN}{2\sqrt{R^2 + \left(\ell/2\right)^2}}.$$

For a long solenoid ($R << \ell$) Thus

$$B = \mu_o \frac{N}{\ell} i$$

cor 2 : To calculate the magnetic field at one end of the solenoid put $a = \ell/2$, therefore

$$B = \mu_o \frac{Ni}{2\ell}\left(\frac{x}{\sqrt{R^2 + x^2}}\right)_0^{\ell}$$

$$\Rightarrow \quad B = \mu_o \frac{Ni}{2\ell} \frac{\ell}{\sqrt{R^2 + \ell^2}}$$

$$\Rightarrow \quad B = \frac{\mu_o iN}{2\sqrt{R^2 + \ell^2}}.$$

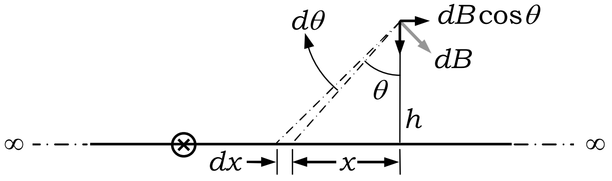
For a long solenoid ($R << \ell$) Thus

$$B = \mu_o \frac{N}{2\ell} i$$

Therefore magnetic field at the centre of a long solenoid is twice of the magnetic field at one of its ends.

Magnetic field near infinite current carrying plane :

If linear current density (current per unit width) in the current carrying conductor is I then considering magnetic field due to element of width dx



$$dB = \frac{\mu_o I dx}{2\pi\left(h/\cos\theta\right)}$$

since $x = h\tan\theta$

hence $dx = h\sec^2\theta d\theta$

putting value of dx we get

$$\Rightarrow \quad dB = \frac{\mu_o Ih\sec^2\theta d\theta}{2\pi\left(h/\cos\theta\right)}$$

only $dB\cos\theta$ component of the magnetic field will survive, hence

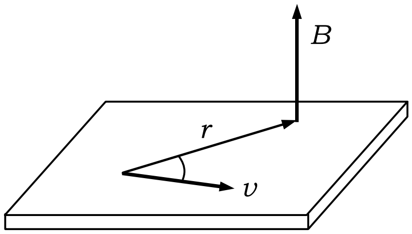
$$\Rightarrow \qquad B = \int dB\cos\theta = \int_{-\pi/2}^{\pi/2} \frac{\mu_o I d\theta}{2\pi}$$

$$\Rightarrow \qquad B = \frac{\mu_o I}{2}$$

<u>Magnetic field of a moving charge</u> :

Instead of current carrying element if we consider magnetic field due to a moving charge at a particular instant, the Biot-Savart's law can be written as

$$d\vec{\mathbf{B}} = \frac{\mu_o}{4\pi}\frac{q\left(\vec{\mathbf{v}}\times\vec{\mathbf{r}}\right)}{r^3}$$ [!]



Magnetic field at the centre due to charge moving in a circle :

Equivalent current

$$i = qf = q\frac{\omega}{2\pi}$$

Hence magnetic field at the centre

$$B = \frac{\mu_o i}{2R} = \frac{\mu_o}{2R}\frac{q\omega}{2\pi}$$

$$\Rightarrow \qquad B = \frac{\mu_o}{4\pi}\frac{q\omega}{R}$$

Same is the result for a charged ring (uniformly or nonuniformly) rotating about its axis.

Magnetic field at the centre due to uniformly charged disk rotating about its axis :

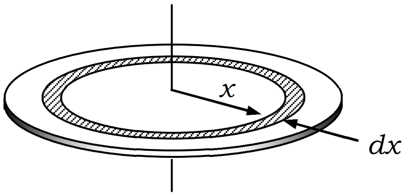
Charge on elemental ring

$$dq = \frac{q}{\pi R^2}2\pi x\left(dx\right) = \frac{2q}{R^2}x\left(dx\right)$$

Equivalent current

$$di = \left(dq\right)f = \frac{\left(dq\right)\omega}{2\pi}$$

$$\Rightarrow \qquad di = \frac{q\omega}{\pi R^2}x\left(dx\right)$$



Magnetic field at the centre due to elemental ring

[!] This result is an approximation to the Biot-Savart's law valid only for $v << c$ (the velocity of light). Since a point charge does not constitute a steady current, where as the Biot-Savart's law which holds only for steady currents does not correctly determine the field of a moving point charge.

$$dB = \frac{\mu_o(di)}{2x} = \frac{\mu_o q\omega}{2\pi R^2 x}x(dx)$$

Hence magnetic field at the centre due to entire disk

$$\Rightarrow \qquad B = \frac{\mu_o q\omega}{2\pi R^2}\int_0^R dx$$

$$\Rightarrow \qquad B = \frac{\mu_o}{2\pi}\frac{q\omega}{R}$$

Magnetic field at the centre due to uniformly charged spherical surface rotating about its diameter :

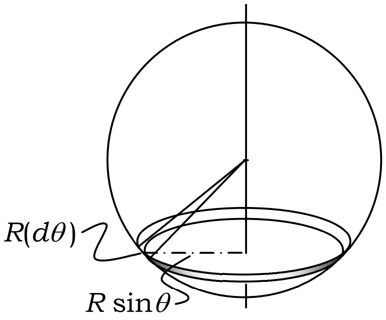
Charge on elemental ring

$$dq = \frac{q}{4\pi R^2}2\pi R\sin\theta(Rd\theta) = \frac{q}{2}\sin\theta(d\theta)$$

Equivalent current

$$di = (dq)f = \frac{(dq)\omega}{2\pi}$$

$$\Rightarrow \qquad di = \frac{q\omega}{4\pi}\sin\theta(d\theta)$$



Magnetic field at the centre due to elemental ring

$$dB = \frac{\mu_o(di)(R\sin\theta)^2}{2\left\{(R\sin\theta)^2 + (R\cos\theta)^2\right\}^{3/2}} = \frac{\mu_o q\omega\sin^3\theta}{8\pi R}d\theta$$

Hence magnetic field at the centre due to entire shell

$$\Rightarrow \qquad B = \frac{\mu_o q\omega}{8\pi R}\int_0^\pi \sin^3\theta d\theta$$

$$\Rightarrow \qquad B = \frac{\mu_o q\omega}{8\pi R}\int_0^\pi \sin\theta\left(1-\cos^2\theta\right)d\theta$$

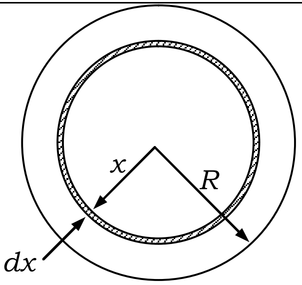
$$\Rightarrow \qquad B = \frac{\mu_o q\omega}{8\pi R}\int_{-1}^1 \left(1-t^2\right)dt$$

$$\Rightarrow \qquad B = \frac{\mu_o q\omega}{8\pi R}\left(t - \frac{t^3}{3}\right)_{-1}^1 = \frac{\mu_o q\omega}{6\pi R}$$

Magnetic field at the centre due to uniformly volume charged sphere rotating about its diameter :

Charge on elemental shell

$$dq = \frac{q}{\frac{4\pi}{3}R^3}4\pi x^2(dx) = \frac{3q}{R^3}x^2(dx)$$



Magnetic field at the centre due to elemental shell

$$dB = \frac{\mu_o (dq)\omega}{6\pi x}$$

$\Rightarrow$ $$dB = \frac{\mu_o q\omega}{2R^3} x(dx)$$

Hence magnetic field at the centre due to entire sphere

$\Rightarrow$ $$B = \frac{\mu_o q\omega}{2R^3}\int_0^R x(dx)$$

$\Rightarrow$ $$B = \frac{\mu_o q\omega}{4R}$$

LORENTZ FORCE

The net force experienced by a charge particle by virtue of its charge is called Lorentz's force. It consists of two components, the electric force and the magnetic force. The force experienced by a moving charge in a magnetic field is $q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$, thus Lorentz's force is

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_e + \vec{\mathbf{F}}_m$$

$\Rightarrow$ $$\vec{\mathbf{F}} = q\vec{\mathbf{E}} + q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

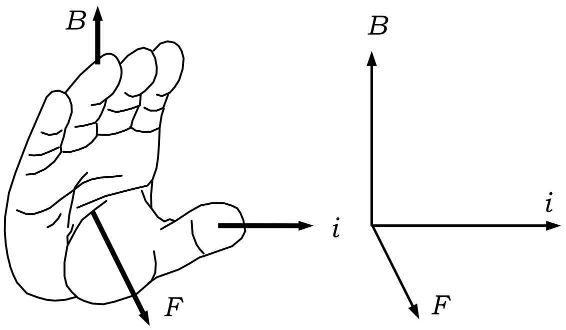
In absence of electric field

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_m = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

The magnitude of force $F = qvB\sin\theta$, where the direction of force is given by the right-hand rule. θ is the angle between the velocity vector and the field vector.

Right hand palm rule :

To find the direction of force on a current carrying conductor (or moving charge), this most suitable rule. Open your right hand palm, set thumb in the direction of motion of positive charge (or opposite to the motion of negative charge), the other fingers in the direction of magnetic field and then the push of the palm indicates the direction of the desired force.



When $\vec{\mathbf{v}}\,||\,\vec{\mathbf{B}}$, $\vec{\mathbf{F}}_m = 0$ and when $\vec{\mathbf{v}} \perp \vec{\mathbf{B}}$, then

$$\vec{\mathbf{F}}_m = \vec{\mathbf{F}}_{max} = qvB.$$

It always acts at right angle to the velocity, thus the charge describes a circle. For a charge particle to describe the circle $\vec{\mathbf{v}} \perp \vec{\mathbf{B}}$, thus

$$\frac{mv^2}{R} = qvB$$

$$\Rightarrow \quad R = \frac{mv}{qB}$$

and time period of motion

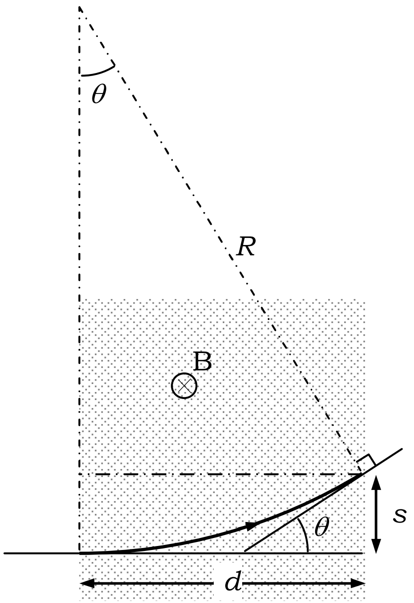
$$T = \frac{2\pi R}{v}$$

$$\Rightarrow \quad T = \frac{2\pi m}{qB}$$

which is independent of speed.

Deflection of charge particle in a uniform magnetic field :

a) Charge particle enters in the region of magnetic field $\perp$ to the region boundary



As it is clear from the figure

If $d < R$ then

$$R\sin\theta = d$$

therefore angle of deflection

$$\Rightarrow \quad \theta = \sin^{-1}\left(d/R\right)$$

The lateral displacement

$$s = R(1-\cos\theta)$$

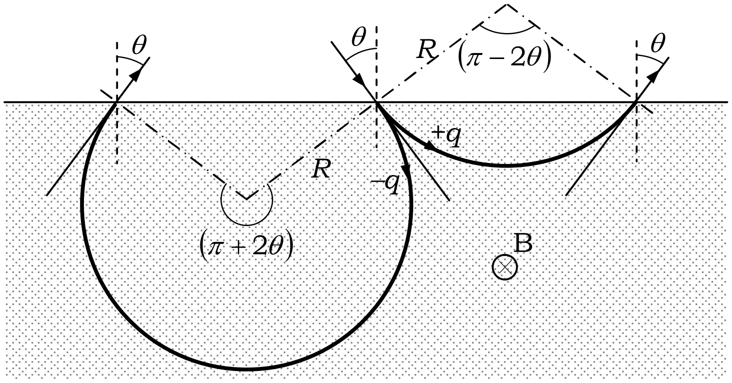
$$s = 2R\sin^2\left(\theta/2\right)$$

If charge particle is moving with very high velocity then the angle of deflection is small thus

$$\Rightarrow \quad s = \frac{R\theta^2}{2} \approx \frac{Rd^2}{2R^2} = \frac{d^2}{2R}$$

If width of region $d > R$ then the angle of deflection $\theta = \pi$ as the particle will describe the semicircle in the region and come back to the same side.

a) Charge particle enters in the region of magnetic field at an angle to the region boundary



From the figure it is evident that the two arcs are complementary
The arc length inside the field for positive charge particle is

$$(\pi - 2\theta)R = (\pi - 2\theta)\frac{mv}{qB}$$

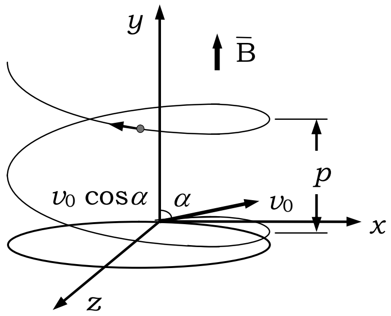
therefore the time for which the particle remains in the field is

$$\frac{Arc}{v} = (\pi - 2\theta)\frac{m}{qB}$$

Similarly for negative charge particle the arc length

$$(\pi + 2\theta)R = (\pi + 2\theta)\frac{mv}{qB}$$

Helical motion :
If velocity vector makes an angle θ with the magnetic field vector then $v_0 \sin\alpha$ (velocity component ⊥ to magnetic field) is responsible for the circular motion and $v_0 \cos\alpha$ moves the particle along the field to generate the helix. Radius of the guiding circle



$R = \dfrac{mv\sin\alpha}{qB}$ and period of revolution

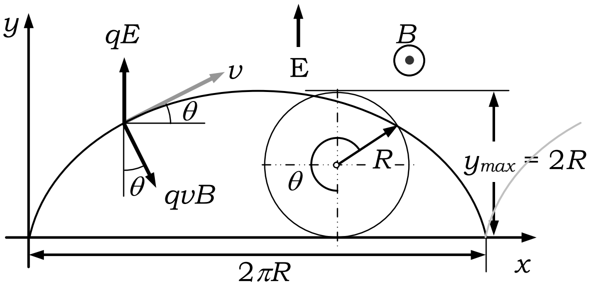
$$T = \frac{2\pi m}{qB}$$

Pitch of the helix (the distance covered in the direction of field in one complete revolution) is

$$p = v_0 \cos\alpha \times T$$

$$p = \frac{2\pi m v_0 \cos\alpha}{qB}$$

Cycloid motion of charge particle :
In mutually perpendicular electric and magnetic fields



At all points on the trajectory

$$v^2 = v_x^{\,2} + v_y^{\,2} \qquad \ldots \qquad (1)$$

as magnetic field does'nt do any work, the work done by electric field on charge particle

$$qEy = \tfrac{1}{2} m v^2$$

$$\Rightarrow \qquad v^2 = \left(\frac{2qE}{m}\right) y \qquad \ldots \qquad (2)$$

considering component of forces along x-axis

$$qvB\sin\theta = m\frac{dv_x}{dt} = m\frac{dv_x}{dy}\frac{dy}{dt} = m\frac{dv_x}{dy} v\sin\theta$$

$$\Rightarrow \qquad qB = m\frac{dv_x}{dy}$$

$$\Rightarrow \qquad dv_x = \frac{qB}{m} dy$$

$$\Rightarrow \qquad v_x = \frac{qB}{m} y + c$$

at $x = 0$ and $y = 0$, v_x, $v_y = 0$ hence $c = 0$. therefore

$$v_x = \frac{qB}{m} y \qquad \ldots \qquad (3)$$

At $y = y_{\max}$ $v_y = 0$ thus $v_x = v$

From equation (2)

$$v^2 = \frac{2qEy_{\max}}{m}$$

and from equation (3)

$$v = \frac{qB}{m} y_{\max}$$

eliminating v, we get

$$y_{\max} = \frac{2mE}{qB^2} = 2R \qquad \ldots \qquad (4)$$

From circular motion of charge particle in uniform magnetic field

$$T = \frac{2\pi m}{qB}$$

$\Rightarrow \quad \omega = \frac{2\pi}{T} = \frac{qB}{m}$ (assume),

and E/B has dimension of velocity hence

$\frac{mE}{qB^2} = R$ (say), then

In equation (1), putting value of v^2 from (2) and v_x from (3)

$$\left(\frac{2qE}{m}\right)y = \left(\frac{qB}{m}\right)^2 y^2 + v_y^{\,2}$$

$$\Rightarrow \quad v_y = \frac{dy}{dt} = \sqrt{\left(\frac{2qE}{m}\right)y - \left(\frac{qB}{m}\right)^2 y^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \left(\frac{qB}{m}\right)\sqrt{\left(\frac{2mE}{qB^2}\right)y - y^2} = \omega\sqrt{2Ry - y^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \omega\sqrt{R^2 - (R-y)^2}$$

$$\Rightarrow \quad \int \frac{dy}{\sqrt{R^2 - (R-y)^2}} = \int \omega dt$$

$$\Rightarrow \quad -\sin^{-1}\left(\frac{R-y}{R}\right) = \omega t + C$$

at $t = 0$ $y = 0$, hence

$$C = -\sin^{-1}(1) = -\left(\pi/2\right)$$

$$1 - y/R = \sin\left(\pi/2 - \omega t\right) = \cos\omega t$$

$$\Rightarrow \quad y = R(1 - \cos\omega t) \qquad \ldots \qquad (5)$$

from (1)

$$v_x = \frac{dx}{dt} = \frac{qB}{m}y = \omega y$$

$$\Rightarrow \quad \frac{dx}{dt} = \omega R(1 - \cos\omega t)$$

$$\Rightarrow \quad \int dx = R\omega\int\{1 - \cos(\omega t)\}dt$$

$$\Rightarrow \quad x = R\omega\left\{t - \frac{\sin(\omega t)}{\omega}\right\} + C'$$

at $t = 0$ $x = 0$, hence $C' = 0$, therefore

$$x = R(\omega t - \sin(\omega t)) \qquad \ldots \qquad (6)$$

Equations (5) and (6) represent cycloid traced by charge particle.

Force experienced by a current carrying conductor in a uniform magnetic field :

If the drift velocity of electrons in the conductor is v_d, area of cross-section is A, length dl and number of free electrons per unit volume is n then force experienced by each electron is

$$f = ev_d B\sin\alpha,$$

therefore force experienced by the conductor is

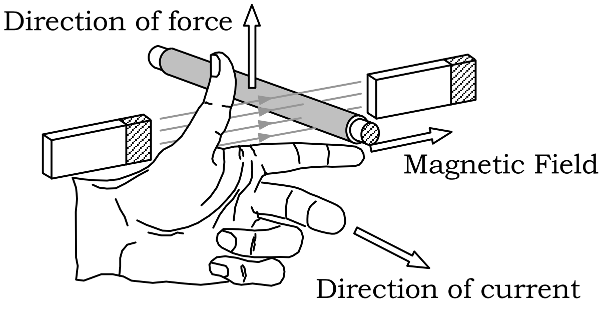
$$dF = n\left(A\,dl\right)ev_d B\sin\alpha$$

$$\Rightarrow \quad dF = \left(neAv_d\right)dl\,B\sin$$

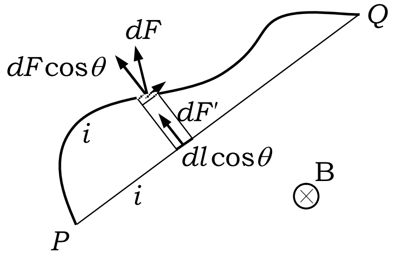
$$\Rightarrow \quad dF = i\,dl\,B\sin\alpha$$

$$\Rightarrow \quad d\vec{\mathbf{F}} = id\vec{\mathbf{l}} \times \vec{\mathbf{B}}$$

Fleming's left hand rule :



Net force on an arbitrary shaped current carrying wire placed in a uniform magnetic field is equal to the force on the straight conductor carrying same current joining the ends of the wire.



Let us consider an element $d\ell$ of the wire. Force on this element is

$dF = Bid\ell$,

component of which normal to PQ is

$iB\ d\ell\cos\theta$

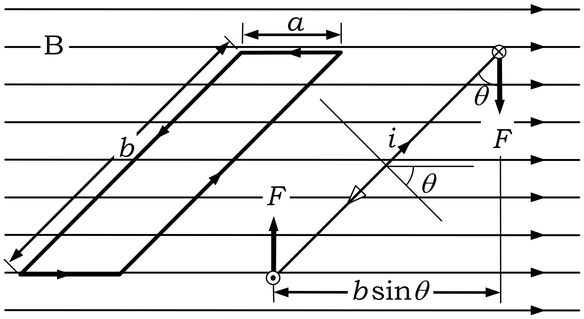
This is same as the force on element projection on PQ which is

$dF\cos\theta = Bi\ d\ell\cos\theta$

$dF\sin\theta$ component on the element $d\ell$ will get canceled when integrated for the entire wire, as the projection length perpendicular to PQ is same for upwards and downwards.

The direct consequence of the above statement is that the net force experienced by a current carrying closed loop in a uniform magnetic field is always zero.

<u>Torque experienced by a current carrying loop placed in a uniform magnetic field</u> :
Let a rectangular loop of sides a and b with n turns caries a current i. The axis of the loop forms an angle θ with the direction of magnetic field. Forces experienced by the arms of length b will be in opposite direction with same line of action therefore they will exactly cancel each other, whereas forces on a will be opposite but there lines of action separated by $b\sin\theta$ thus torque experienced by the loop is



$$\tau = Fb\sin\theta = (niaB)b\sin\theta$$

$\Rightarrow$ $\tau = niAB\sin\theta$

Where A is the area of rectangular loop.

Comparing this with the expression for a bar magnet placed in a uniform magnetic field we get

$M = niA$ (magnetic moment of the loop carrying current)

Hence a current carrying loop behaves as a bar magnet placed on its axis whose magnetic moment is given by

$M = niA$

It is valid for current carrying loop of any shape.

Representative example 1 :

A uniform wire of length ℓ is bent to form a coil of n turns. Find the strength of magnetic field at its centre and total magnetic moment associated with the coil.

Solution :

Radius of the coil is R then

$$n(2\pi R) = \ell$$

$\Rightarrow$ $R = \frac{\ell}{2\pi n}$

Magnetic field at its centre

$$B = \frac{\mu_o ni}{2R} = \frac{\mu_o ni}{2\ell}2\pi n$$

$\Rightarrow$ $B = n^2\frac{\mu_o i}{\ell}$ hence $(B \propto n^2)$

Magnetic moment associated with the coil

$$M = niA = ni(\pi R^2)$$

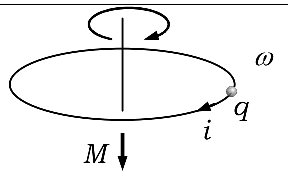
$\Rightarrow$ $M = ni\pi\left(\frac{\ell}{2\pi n}\right)^2$

$\Rightarrow$ $M = \frac{i\ell^2}{4\pi n}$ hence $\left(M \propto \frac{1}{n}\right)$

Magnetic moment of char ge moving in a circle :

Equivalent current

$$i = qf = q\frac{\omega}{2\pi}$$



Hence magnetic moment

$$M = i\left(\pi R^2\right)$$

$$\Rightarrow \quad M = \frac{q\omega}{2\pi}\left(\pi R^2\right) = \frac{q\omega R^2}{2}$$

Same is the result for a charged ring (uniformly or non-uniformly) rotating about its axis.

Magnetic moment of uniformly charged disk rotating about its axis :

Charge on elemental ring

$$dq = \frac{q}{\pi R^2} 2\pi x(dx) = \frac{2q}{R^2} x(dx)$$

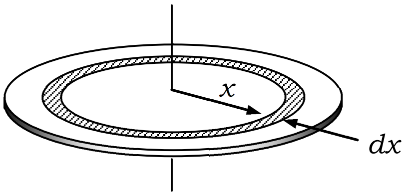
Equivalent current

$$di = (dq) f = \frac{(dq)\omega}{2\pi}$$

$$\Rightarrow \quad di = \frac{q\omega}{\pi R^2} x(dx)$$

Magnetic moment of elemental ring

$$dM = (di)\pi x^2 = \frac{q\omega}{R^2} x^3 (dx)$$



Hence magnetic moment of entire disk

$$\Rightarrow \quad M = \frac{q\omega}{R^2}\int_0^R x^3 (dx)$$

$$\Rightarrow \quad M = \frac{q\omega R^2}{4}$$

Magnetic moment of uniformly charged spherical surface rotating about its diameter :

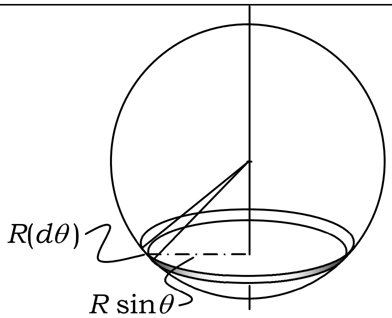
Charge on elemental ring

$$dq = \frac{q}{4\pi R^2} 2\pi R \sin\theta (R d\theta) = \frac{q}{2}\sin\theta (d\theta)$$

Equivalent current

$$di = (dq) f = \frac{(dq)\omega}{2\pi}$$

$$\Rightarrow \quad di = \frac{q\omega}{4\pi}\sin\theta (d\theta)$$



Magnetic moment of elemental ring

$$dM = (di)\pi(R\sin\theta)^2 = \frac{q\omega R^2\sin^3\theta}{4}d\theta$$

Hence magnetic moment of entire shell

$$\Rightarrow \quad dM = \frac{q\omega R^2}{4}\int_0^{\pi}\sin^3\theta d\theta$$

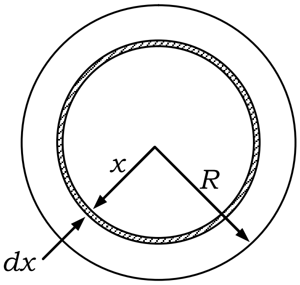
$$\Rightarrow \quad M = \frac{q\omega R^2}{4}\int_0^{\pi}\sin\theta(1-\cos^2\theta)d\theta$$

$$\Rightarrow \quad M = \frac{q\omega R^2}{4}\int_{-1}^{1}(1-t^2)dt$$

$$\Rightarrow \quad M = \frac{q\omega R^2}{4}\left(t-\frac{t^3}{3}\right)_{-1}^{1} = \frac{q\omega R^2}{3}$$

Magnetic moment of uniformly volume charged sphere rotating about its diameter
Charge on elemental shell

$$dq = \frac{q}{\frac{4\pi}{3}R^3}4\pi x^2(dx) = \frac{3q}{R^3}x^2(dx)$$



Magnetic moment of elemental shell

$$dM = \frac{(dq)\omega x^2}{3}$$

$$\Rightarrow \quad dM = \frac{q\omega}{R^3}x^4(dx)$$

Hence magnetic moment of entire sphere

$$\Rightarrow \quad M = \frac{q\omega}{R^3}\int_0^{R}x^4(dx)$$

$$\Rightarrow \quad M = \frac{q\omega R^2}{5}$$

Note :

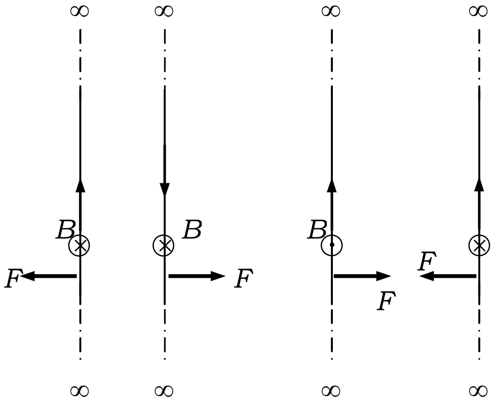
If a uniformly charged object (charge distributed following mass distribution in the object) is rotated about an axis, the ratio of magnetic moment and mechanical moment (angular momentum) of the object remains constant given by

$$M/L = \frac{M}{I\omega} = \frac{Q}{2m}$$

where Q is total charge on the object and m is the mass of the object.

Force between two infinite parallel current carrying wires :

Let the wires carrying currents i_1 and i_2 respectively are separated by a distance r. Taking an arbitrary point P on the second wire, the magnetic field produced by the first wire on the position of the second wire is



$B_{21} = \frac{\mu_o i_1}{2\pi r}$ perpendicular to the wire. Therefore the force on and element l of the second wire is

$$F_{21} = i_2 \frac{\mu_o i_1}{2\pi r}\ell = \frac{\mu_o i_1 i_2}{2\pi r}\ell$$, thus force per unit length

$$\frac{F_{21}}{\ell} = \frac{\mu_o i_1 i_2}{2\pi r}.$$

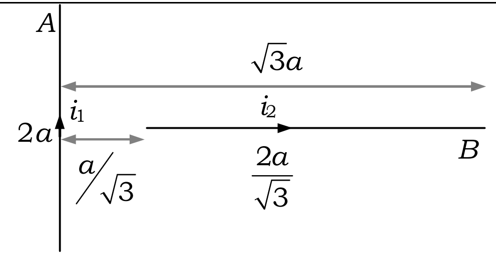
We note that wires carrying current in the same direction attract (Verify using Right hand palm rule).

Ampere :

When two parallel, long, straight, current carrying wires of negligible cross-section carrying same current and separated by a distance of 1 m apply a force of 2×10^{-7} N per unit length on each other, the current flowing in them is one ampere.

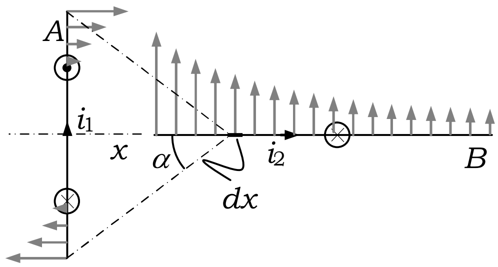
Representative example 2 :

There are two coplanar, mutually perpendicular, current carrying segments of length $2a$ and $\frac{2a}{\sqrt{3}}$ carrying currents i_1 and i_2 respectively as shown in figure. Find the force on conductor A due to B and the force on conductor B due to A.



Solution :

Force on *A* due to *B* is zero as the magnetic field on upper half of conductor *A* due *B* is out of the plane where as on the lower half it is in to the plane, hence due to symmetry the two halves of conductor *A* experience equal and opposite forces.



Magnetic field on *B* due to *A* at a distance *x* from A is

$$B=\frac{\mu_o i_1}{2\pi x}\sin\alpha$$

Force on small element of *B*

$$dF=i_2Bdx=\frac{\mu_o i_1 i_2}{2\pi x}\sin\alpha\,dx$$

$$\Rightarrow \quad F=\frac{\mu_o i_1 i_2}{2\pi}\int_{\pi/6}^{\pi/3}\frac{a}{x\sqrt{a^2+x^2}}dx$$

$$\Rightarrow \quad F=\frac{\mu_o i_1 i_2}{2\pi}\ln\left(\tan\alpha/2\right)_{\pi/6}^{\pi/3}$$

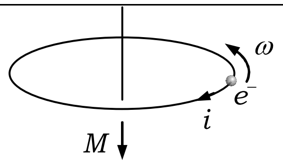
$$\Rightarrow \quad F=\frac{\mu_o i_1 i_2}{2\pi}\ln\left(\frac{\tan\pi/6}{\tan\pi/12}\right)$$

Note that the force on *A* due to *B* is zero where as force on *B* due to *A* is nonzero. As they are not the isolated objects hence Newton's 3rd law is not valid on this system.

MAGNETIC PROPERTIES OF MATERIALS

Atomic magnetic dipole :

In every atom electrons revolve around nucleus forming current loops with associated current



$$i = \frac{e\omega}{2\pi}$$

Which result in magnetic dipole with dipole-moment

$$M = iA = \frac{e\omega}{2\pi}\pi r^2 = \frac{e\omega r^2}{2}$$

where ω is the orbital velocity of electron in its orbit and m the mass of electron. According to 3rd postulate of Bohr's theory angular momentum of electron in its stable orbit should be complete multiple of $h/2\pi$, therefore

$$mvr = m\omega r^2 = n\frac{h}{2\pi}$$

Therefore magnetic dipole moment of the electron in its orbit is

$M = \frac{e}{2}n\frac{h}{2\pi m} = n\frac{eh}{4\pi m} = n\mu_B$ where μ_B is a constant as e, h and m all are constants, thus

$$\mu_B = \frac{eh}{4\pi m} = 9.27 \times 10^{-24}\ \text{A m}^2$$

Here μ_B is called Bohr's magneton. It is the magnetic dipole moment associated with hydrogen atom due to revolution of electron around proton (nucleus) in its ground state (first orbit).

Magnetic induction, Magnetic flux density or Magnetic field (B) :
When a magnetic substance is placed inside an external magnetizing field the substance gets magnetized. Magnetism thus produced inside the substance is called induced magnetism and the phenomenon is called magnetic induction. This is also known as magnetic flux density or magnetic field and denoted by $\vec{\mathbf{B}}$.
Unit of magnetic induction is tesla (T) or weber m^{-2}. In SI system $T = J\ A^{-1}\ m^{-2}$

$$[B] = [MT^{-2}A^{-1}]$$

Magnetic Intensity or magnetizing force (H) :
The field which induces magnetic properties in a substance, when placed in it, is called magnetizing field. The strength of Magnetizing field, Magnetizing force or Magnetic intensity at a point is defined as the force experienced by a unit north pole placed at that point.
Unit of magnetic intensity is $N\ Wb^{-1}$. In SI system it is $A\ m^{-1}$.

$$[H] = [L^{-1}A]$$

Magnetic permeability (μ) :
Magnetic permeability of a substance is the ability of the substance to let the magnetic lines of forces pass through it. It is represented by μ.

Numerically it is defined as the ratio of the magnetic induction of the given material to the magnetizing field.

$$\mu = B/H$$

Unit of magnetic permeability is $N\ A^{-2}$ or $Wb\ A^{-1}\ m^{-1}$.

$[\mu] = [MLT^{-2}A^{-2}]$

Intensity of magnetization (I) :

It is the extent to which a material can be magnetized when placed in a magnetizing field. Numerically it is defined as the magnetic moment developed per unit volume of the sample when placed in a magnetizing field.

$$I = \frac{\text{Magnetic moment}}{\text{Volume}} = \frac{p_m}{V}$$

For bar magnet with pole strength m, length d and area of cross-section A

$$I = \frac{p_m}{V} = \frac{md}{Ad} = m/A$$

Therefore the intensity of magnetization of the material may also be defined as the pole-strength developed per unit area of cross-section of the given material.

Unit of intensity of magnetization A m^{-1} it is same as that of magnetic intensity.

$[I] = [L^{-1}A]$

Magnetic Susceptibility (χ) :

Magnetic susceptibility of a material is that property of material which expresses the ease of magnetization of the material when it is exposed to magnetizing field. Numerically it is defined as the ratio of intensity of magnetization (I) to the magnetizing field (H).

$$\chi = I/H$$

It is a dimensionless quantity.

Important relations

1. Relation between $\vec{\mathbf{B}}$, $\vec{\mathbf{H}}$ and $\vec{\mathbf{I}}$:

When a magnetic material is placed in a magnetizing field of strength H, the material is magnetized (acquires induced magnetism) then the magnetic induction B developed inside the material is the sum of magnetic induction caused by magnetizing field in free space and the magnetic induction caused by the magnetized material

$$B = B_o + B_m$$

$$\Rightarrow \quad B = \mu_o H + \mu_o I$$

$$\Rightarrow \quad B = \mu_o (H + I)$$

2. Relation between μ_r and χ :

From the above relation

$$B = \mu_o (H + I)$$

$$\Rightarrow \quad \mu H = \mu_o (H + I)$$

$$\Rightarrow \quad \mu/\mu_o = \left(1 + I/H\right)$$

$$\Rightarrow \quad \mu_r = (1 + \chi)$$

Classification of Magnetic Materials :

Diamagnetic Substances :

When exposed to external magnetic field these substances get feebly magnetized opposite to the magnetizing field. A freely suspended rod of diamagnetic material in an external magnetic field stays perpendicular to the magnetic field. Permeability of diamagnetic substances is always less then 1, therefore susceptibility of the diamagnetic material is negative and it is independent of temperature.

Paramagnetic Substances :

When exposed to external magnetic field these substances get feebly magnetized in the direction of magnetizing field. A freely suspended rod of paramagnetic material in an external magnetic field stays aligned with its larger axis to the magnetic field. Permeability of paramagnetic substances is always greater then 1, therefore susceptibility of the paramagnetic material has a small positive value.

Ferromagnetic Substances :

When exposed to external magnetic field these substances get strongly magnetized in the direction of magnetizing field. A freely suspended rod of ferromagnetic material in an external magnetic field stays aligned with its larger axis to the magnetic field. Permeability of ferromagnetic substances is very large compared unity, therefore susceptibility of the ferromagnetic material has a large positive value.

Curie's Law of magnetism :

According to Curie the intensity of magnetization (I) of the paramagnetic materials is directly proportional to magnetic intensity of the magnetizing field and it is inversely proportional to the absolute temperature of the sample.

$I \propto H$ and

$$I \propto 1/T$$

Therefore

$$I = C\left(H/T\right)$$

Where C is the constant of proportionality and is called Curie's constant. Now

$$I/H = C/T$$

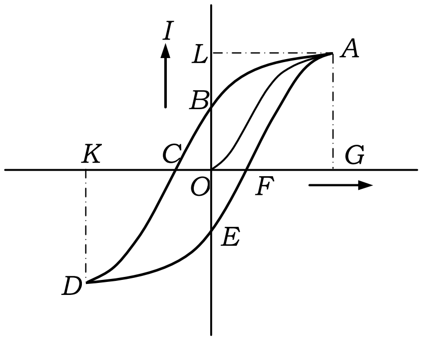
Therefore

$$\chi \propto 1/T$$

Hence the magnetic susceptibility of paramagnetic materials is inversely proportional to its absolute temperature. This statement is called Curie's law in magnetism.

Hysteresis :

When the sample of a magnetic material is exposed to an external magnetizing force the sample is magnetized following OA on adjacent diagram. When the external magnetizing force is reduced the sample is demagnetized following AB on the adjacent diagram. Therefore sample will have residual magnetization left even when the external magnetizing force is reduced to zero. *The value of the intensity of magnetization of material when the magnetizing force is reduced to zero is called Retentivity or Residual magnetism of the material.* When the external magnetizing force is reversed the intensity of magnetization of material becomes zero at C. *The external magnetizing force in opposite direction to reduce down the intensity of magnetization of the sample is called the Coercivity of the sample.*



Cyclotron :

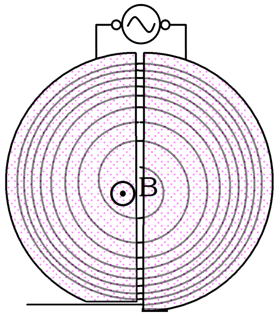
In 1932 Lawrence and Livingston at University of California, devised an instrument to accelerate positively charged particles i.e. protons, deuterons, α-particles etc to high velocity. These high energy particles are then used to carryout various experiments related to subatomic particles.

Principle :

Charge particles moving perpendicular to magnetic field move on a circular path where as electric field accelerates them to change their energy.

Construction :

The basic design of a cyclotron consists of a pair of hollow D-shaped conducting chambers kept horizontal with diametric edges parallel and separated from each other by a small separation. A high frequency alternating potential source is connected between the D's to generate electric field in the gap between them. The hole arrangement is electrically shielded (enclosed in a metallic container) and then placed between the poles of strong electromagnets which produces strong and uniform, vertical magnetic field perpendicular to the plane of D's. Positive ions to be accelerated are produced in a separate mechanism and injected in the gap between the D's at *O*.



Working : Positively charged ions are accelerated towards negative plate and due to magnetic field they move in a circle. The time in which they complete the same-circle the polarity of potential between the D's changes and they are further accelerated to increase their velocity a little. This increases their radius of revolution also by a little amount but since their time period of rotation is independent their velocity hence they keep moving with same frequency and alternating potential applied between D's keep increasing their velocity. When the radius of their revolution becomes large enough they are deflected out of the D's with the help of deflecting potential applied at the edge of one of the D's.

Theory :

For positive ions of mass m, charge q, moving with a velocity v in a uniform magnetic field B, the centripetal force is provided by force due to magnetic field, hence radius of circular orbit

$$qvB = \frac{mv^2}{r}$$

$$\Rightarrow \quad r = \frac{mv}{qB}$$

As velocity of the ions increase the radius of the ions also increase.

Time period of revolution of the ions is

$$T = \frac{2\pi r}{v}$$

$$\Rightarrow \quad T = \frac{2\pi mv}{vqB} = \frac{2\pi m}{qB}$$

Which is independent velocity.

Maximum kinetic energy associated with the ions when the ions reach to the radius approximately equal to the radius of the D's.

$$R_D = \frac{mv_m}{qB}$$

$$\Rightarrow \quad v_m = \frac{qBR_D}{m}$$

Therefore maximum kinetic energy of ions

$$E_{k(m)} = \frac{1}{2}mv_m{}^2 = \frac{q^2B^2R_D{}^2}{2m}$$

Limitations :

i) It can not accelerate uncharged particles like neutrons.

ii) It can not accelerate electrons (very light particles) as they gain very high velocity while moving through small potential difference and go out of phase of applied potential difference.

iii) It can not accelerate charge particles beyond limit (relativistic reasons).

Magnetic field/moment of various, rotating charge distributions

	Object	Magnetic field	Magnetic Moment	
1.	Ring	$\frac{\mu_o q\omega}{4\pi R}$	$\frac{q\omega R^2}{2}$	
2.	Disk	$\frac{\mu_o q\omega}{2\pi R}$	$\frac{q\omega R^2}{4}$	
3.	Spherical Shell	$\frac{\mu_o q\omega}{6\pi R}$	$\frac{q\omega R^2}{3}$	
4.	Solid sphere	$\frac{\mu_o q\omega}{4\pi R}$	$\frac{q\omega R^2}{5}$	

ELECTROMAGNETIC INDUCTION

Magnetic flux :

Magnetic flux bounded by a surface is defined as the total number of magnetic lines of force crossing the surface from one side to the other. From the definition of the magnetic field, magnetic flux is the dot product of the magnetic field strength vector and the area vector.

$$d\varphi = \vec{\mathbf{B}}.d\vec{\mathbf{s}}$$

$$\Rightarrow \quad \varphi = \int \vec{\mathbf{B}}.d\vec{\mathbf{s}}$$

Farade's Law of electromagnetic induction :

The rate of change of magnetic flux bounded by the area of closed conducting loop is equal to the emf induced in the loop.

$\mathcal{E} = -\frac{d\varphi}{dt}$, As we know that the electric field strength is the gradient of potential, thus

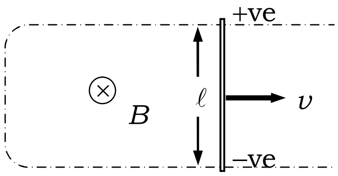
$$\mathcal{E} = \oint \vec{\mathbf{E}}.d\vec{\mathbf{l}} = -\frac{d\varphi}{dt}.$$

The minus sign was introduced by Lenz.

Lenz's Law :
The direction of the induced emf is such that it opposes the very cause of its production.

Origin of induced emf, the Lorentz force :
Let us consider a straight conductor of length ℓ moving perpendicular to a constant and uniform magnetic field B with a velocity v. The force experienced by a free electron inside the conductor

is qvB downwards. The electrons will start accumulating at the lower end, creating a potential difference of $\mathcal{E}$. Maximum value of this potential will be attained when a free electron in the conductor experiences same electric force in the induced electric field upwards as the downwards magnetic force.



$$eE = evB$$

$$\Rightarrow \quad E = \frac{\mathcal{E}}{\ell} = Bv$$

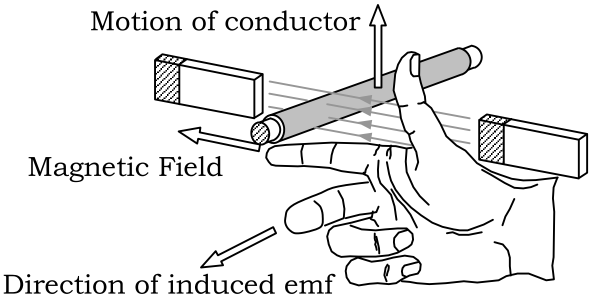
$$\Rightarrow \quad \mathcal{E} = Bv\ell\text{, but}$$

$$v = \frac{dx}{dt} \text{ therefore}$$

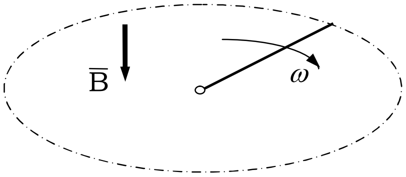
$$\mathcal{E} = B\ell\frac{dx}{dt} = B\frac{dA}{dt} = \frac{d\varphi}{dt}$$

To explain the negative sign put by the Lanz it can be argued that the conductor moving with velocity v is increasing the magnetic flux by increasing the area of the loop while the induced current is producing the magnetic field opposite to already present magnetic field in the loop which is trying to reduce the magnetic flux bounded by the loop. Thus the direction of the induced emf is opposite to the cause of its production.

Fleming's right hand rule :



Induced emf between the ends of a rod :
Let a conducting rod of length ℓ is rotating about an axis perpendicular to the rod and passing through its end. Let there be a magnetic field B is present perpendicular to the plane of rotation. Consider a free electron at a distance x from the axis of rotation, if electric field established by the induced emf is E then net force on electron is



$$-eE \pm e(x\omega)B = mx\omega^2$$

$$\Rightarrow \quad -E = \frac{dV}{dx} = x\left(\frac{m\omega^2}{e} \mp \omega B\right)$$

$$\Rightarrow \quad \mathcal{E} = \int dV = \left(\frac{m\omega^2}{e} \mp \omega B\right)\left(\frac{x^2}{2}\right)_0^{\ell}$$

$$\Rightarrow \quad \mathcal{E} = \left(\frac{m\omega^2}{e} \mp \omega B\right)\frac{\ell^2}{2}$$

As mass of the electron is too small compared to electronic charge, the first term in the above expression is negligible. Thus

$$\mathcal{E} \approx \frac{B\omega\ell^2}{2}$$

Amount of charge flow due to change in flux $\Delta\varphi$ though the loop :

If the magnetic flux bounded by a closed loop with loop resistance R, is φ_1 initially which changes to φ_2 then for a small change in flux $d\varphi$ in very small time interval dt

$$i = \frac{\mathcal{E}}{R} = -\frac{1}{R}\frac{d\varphi}{dt} = \frac{dq}{dt}$$

$$\Rightarrow \quad dq = -\frac{d\varphi}{R}$$

$$\Rightarrow \quad q = \frac{\varphi_1 - \varphi_2}{R}$$

$$\Rightarrow \quad q = \frac{\Delta\varphi}{R}$$

Work done in changing the flux by $\Delta\varphi$ though a current carrying loop, maintaining the current constant :

If the magnetic flux bounded by a closed loop with loop current i, is φ_1 initially which changes to φ_2 then for a small change in flux $d\varphi$ in very small time interval dt

$$dw = -i\mathcal{E}\,dt$$

$$\Rightarrow \quad dw = -i\frac{d\varphi}{dt}dt = id\varphi$$

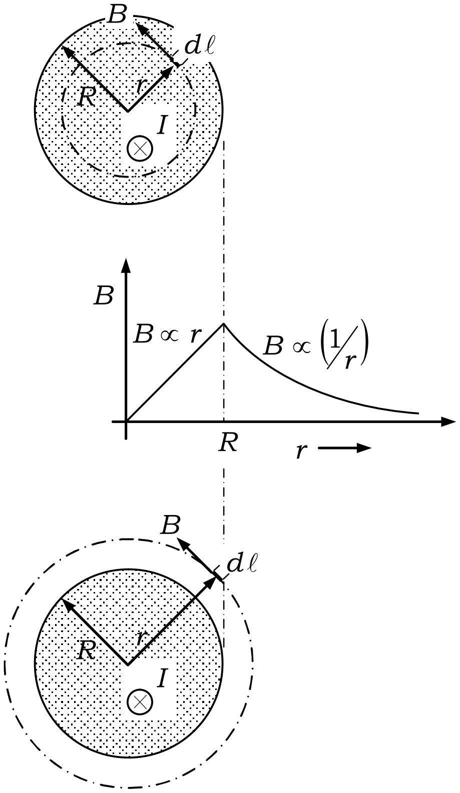
$$\Rightarrow \quad w = i(\varphi_1 - \varphi_2)$$

$$\Rightarrow \quad w = i\Delta\varphi$$

Ampere's circuital Law :

If a total current i, either concentrated or distributed, pierces through (crosses) a closed path, in that case $\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_o i$. Here B is the magnetic field (due to all sources inside as well as out side the loop) at the points on the loop. $\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}}$ is the line integral around the closed path.

<u>Magnetic field inside and outside a long, uniform straight cylindrical wire</u> :
Let the wire carries a current I then consider the concentric circle of radius r in the cross section of the wire. For $r < R$, the amount of piercing current is



$$\frac{I}{\pi R^2}\left(\pi r^2\right) = I\left(r^2/R^2\right)$$

hence from Ampere's law

$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_o I\frac{r^2}{R^2}$$

as due to symmetry B is constant in magnitude and tangential every where therefore

$$\Rightarrow \quad B\left(2\pi r\right) = \mu_o I\frac{r^2}{R^2}$$

$$\Rightarrow \quad B = \frac{\mu_o}{2\pi}\frac{I}{R^2}r$$

Since B is directly proportional to r therefore at the axis of wire the magnetic field is zero.

For $r > R$, the amount of piercing current remains constant to I hence from Ampere's law

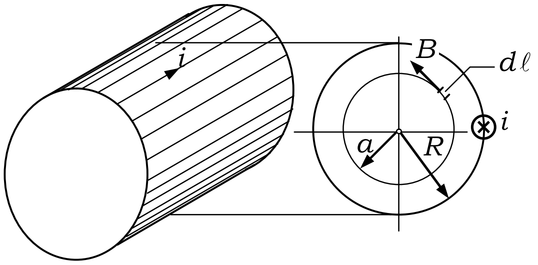
$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_o I$$

as due to symmetry B is constant in magnitude and tangential every where therefore

$$\Rightarrow \quad B\left(2\pi r\right) = \mu_o I$$

$$\Rightarrow \quad B = \frac{\mu_o}{2\pi}\frac{I}{r}$$

Magnetic field inside a long, thin walled conducting pipe carrying current :
Consider a coaxial circle of radius a ($< R$). Due to symmetry, at every point on it the magnetic field should same in magnitude and tangential in direction. Since the current piercing this circle is zero hence from Ampear's law

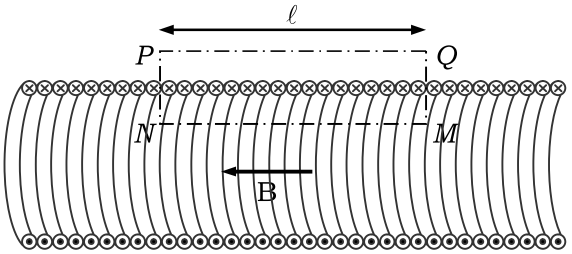


$$\oint \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \mu_o(0) = 0$$

$$\Rightarrow \quad B = 0$$

Therefore at any general point inside a long, uniform, thin pipe carrying a current i is zero.

Magnetic field inside a long straight solenoid :
Consider a rectangle $PQMN$ in the symmetry plane of the solenoid as shown in figure



As magnetic field outside the solenoid is negligibly small (theoretically it should be zero), hence from Ampere's law

$$\oint_{PQMN} \vec{\mathbf{B}}.d\vec{\mathbf{l}}$$

$$\Rightarrow \quad \int_{PQ} \vec{\mathbf{B}}.d\vec{\mathbf{l}} + \int_{QM} \vec{\mathbf{B}}.d\vec{\mathbf{l}} + \int_{MN} \vec{\mathbf{B}}.d\vec{\mathbf{l}} + \int_{NP} \vec{\mathbf{B}}.d\vec{\mathbf{l}}$$

At PQ magnetic field is zero, at QM and NP it is perpendicular to respective arms of loop and at MN, magnetic field has uniform magnitude and tangential to MN, then

$$\oint_{PQMN} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = 0 + 0 + \int_{MN} B\,d\ell \cos 0 + 0$$

$$\Rightarrow \quad \oint_{PQMN} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = B\ell$$

If the number of turns per unit length of the solenoid is n then amount of current piercing through the rectangular loop is

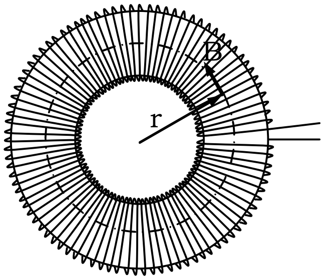
$$ni\ell$$

$$\Rightarrow \quad \oint_{PQMN} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = B\ell = \mu_o(ni\ell)$$

$$\Rightarrow \quad B = \mu_o ni$$

Magnetic field inside a toroidal solenoid :
Consider a circle of radius r the symmetry plane of the toroid as shown in figure hence from Ampere's law

$$\oint_{circle} \vec{\mathbf{B}}.d\vec{\mathbf{l}}$$



Due to symmetry magnetic field is same at all points on the circumference of the circle and tangential to the circle hence

$$\oint_{circle} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = \oint_{circle} B.d\ell \cos 0$$

$$\Rightarrow \quad \oint_{circle} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = B\oint_{circle} d\ell$$

$$\Rightarrow \quad \oint_{circle} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = B(2\pi r)$$

If the total number of turns in the toroid is N then amount of current piercing through the circular loop is

Ni

$$\Rightarrow \quad \oint_{circle} \vec{\mathbf{B}}.d\vec{\mathbf{l}} = B(2\pi r) = \mu_o(Ni)$$

$$\Rightarrow \quad B = \frac{\mu_o Ni}{2\pi r} = \mu_o ni$$

Where n is number of turns per unit length of the toroid.

<u>TIME VARYING MAGNETIC FIELD</u> :

Consider the concentric circle of radius r in a cylindrical region of magnetic field changing with time. From Farade's law of electromagnetic induction

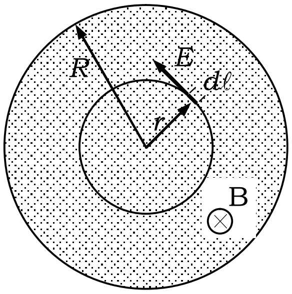
For $r < R$

$$\varphi = \pi r^2 B$$

$$\mathcal{E} = \frac{d\varphi}{dt} = \pi r^2 \frac{dB}{dt}$$

but

$$\mathcal{E} = \oint \mathbf{E}.d\mathbf{l} = E2\pi r$$



therefore

$$E = \frac{r}{2}\frac{dB}{dt}$$

this electric field is tangential to the circle hence non-conservative.

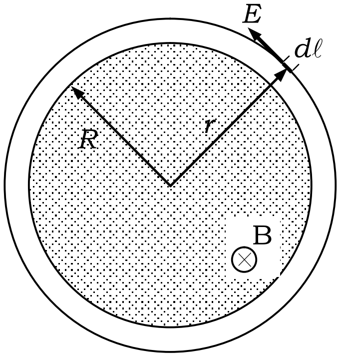
For $r > R$

$$\varphi = \pi R^2 B$$

$$\mathcal{E} = \frac{d\varphi}{dt} = \pi R^2 \frac{dB}{dt}$$

but

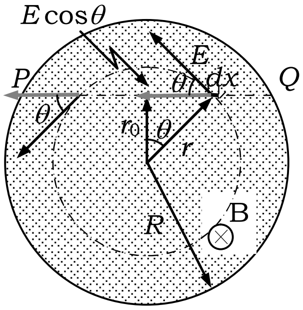
$$\mathcal{E} = \oint \mathbf{E}.d\mathbf{l} = E2\pi r$$



therefore

$$E = \frac{R^2}{2r}\frac{dB}{dt}$$

this electric field is tangential to the circle hence non-conservative.



The potential difference across the cord *PQ* at any instant is

$$V = \int \mathbf{E}.d\mathbf{x}$$

$$\Rightarrow \quad V = \int E\cos\theta dx$$

$$\Rightarrow \quad V = \int \frac{r}{2}\frac{dB}{dt}\left(\frac{r_0}{r}\right)dx$$ if, B is not changing with x (B is uniform over entire area)

$$\Rightarrow \quad V = \frac{r_0}{2}\frac{dB}{dt}\int dx$$

$$\Rightarrow \quad V = \frac{r_0}{2}\frac{dB}{dt}2\sqrt{R^2 - {r_0}^2}$$

$$\Rightarrow \quad V = r_0\sqrt{R^2 - {r_0}^2}\,\frac{dB}{dt}$$

Eddy currents :

Discovered by Foucault in 1895, when magnetic flux bound by a conductor is changed electric currents in the form of concentric closed loops are induced in the body of the conductor called eddy current. The direction of eddy current is given by lenz's law. Eddy currents oppose the relative motion and involve loss of energy in the form of heat.

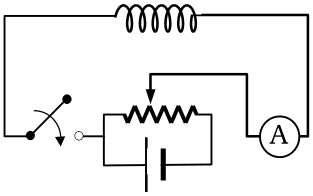
Applications :

a) Eddy currents are used in electromagnetic damping and breaks.
b) Heating produced by eddy currents is used to derive induction furness.
c) Induction motion also works on the principal of eddy current.

To reduce loss of energy due to eddy currents the resistance of material is increased by converting it into thin plates insulated from each other, like transformer core, core of choke coil etc.

Self Induction :

The phenomenon of generation of induced emf in a coil due to change in flux bounded by the coil because of its own magnetic field (produced by the current flowing in the coil it self) is called self induction.



If the total number of turns in the coil are N then total flux bounded by the coil is

$$N\varphi = NA\left(\mu_o \frac{N}{\ell} i\right) = \left(\frac{\mu_o N^2 A}{\ell}\right) i$$

⇒ $N\varphi \propto i$

⇒ $N\varphi = Li$, therefore

$$L = \frac{\mu_o N^2 A}{\ell}$$

where L is called the coefficient of self induction of the coil

Energy stored in the inductor :

The work done in charging the inductor is

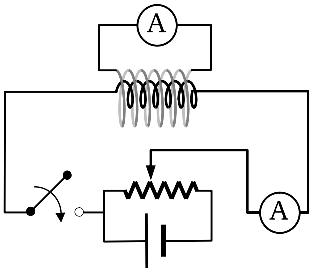
$$w = \int i\mathcal{E}$$

⇒ $U = \int iL \frac{di}{dt}$

⇒ $U = \frac{1}{2} Li^2$, where $\mathcal{E}$ is induced emf.

Mutual Induction :

The phenomenon of generation of induced emf in a coil (called secondary) due to change in flux bounded by the coil because of the magnetic field produced by the current flowing in another coil inside it (called primary) or otherwise, is called mutual induction.



If the total number of turns in the primary coil are N_1 and secondary coil is N_2 then total flux bounded by the primary coil is

$$N_2\varphi_2 = N_2 A_1\left(\mu_o \frac{N_1}{\ell_1} i_1\right) = \left(\frac{\mu_o N_1 N_2 A_1}{\ell_1}\right) i_1$$

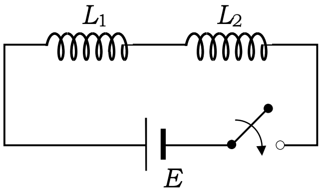
$\Rightarrow$ $N_2\varphi_2 \propto i_1$

$\Rightarrow$ $N_2\varphi_2 = M i_1$, therefore

$$M = \frac{\mu_o N_1 N_2 A_1}{\ell_1}$$

where M is called the coefficient of mutual induction between the coils.

Combination of inductors :
Inductors in series :
In both the inductors current is same hence



$$E = \mathcal{E}_1 + \mathcal{E}_2$$

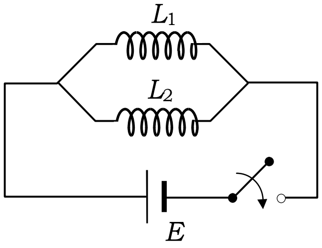
$\Rightarrow$ $L_{eq}\frac{di}{dt} = L_1\frac{di}{dt} + L_2\frac{di}{dt}$

$\Rightarrow$ $L_{eq} = L_1 + L_2$

Inductors in parallel :
Total current is sum of currents in both the inductors hence

$$I = i_1 + i_2$$

$\Rightarrow$ $\frac{dI}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt}$



$\Rightarrow$ $\frac{E}{L_{eq}} = \frac{E}{L_1} + \frac{E}{L_2}$

$\Rightarrow$ $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2}$

Coupling factor :
It is the measure of magnetic linkage between the two coils. If the coefficients of self induction of the two coils are L_1 and L_2 respectively, then the coupling factor between them is

$$K = \sqrt{\frac{M}{L_1 L_2}}$$

where M is coefficient of mutual induction between the two coils. Coupling factor is always less than unity (at the most it can be one but in all practical situations it is less than one).

<u>Transformer</u> :

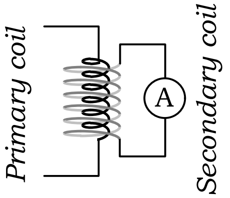
It is a device used with alternating current to increase or decrease the ac voltage by a required factor.

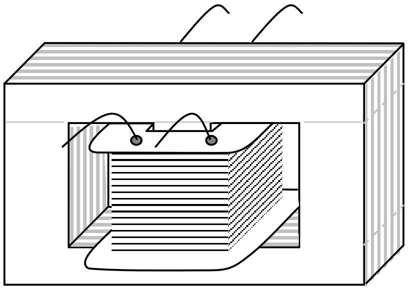
Principle :

It is based on the principle of mutual induction between two coils, primary where input voltage is applied and the secondary where output is received.

Construction :

It consists of a rectangular soft iron core made of laminated, well insulated thin iron sheets. Two coils are wound one over the other, a primary and a secondary such that magnetic flux linked with one turn of primary is same as the magnetic flux linked with one turn of secondary.





Working :

When a current changing with time (alternating current) passes through the primary coil total flux linked with it changes with time because of which total flux linked with secondary also changes with time, which induces emf in the secondary and hence induced current in the output circuit. If the number of turns in the secondary coil are greater than the number of turns in primary then total flux linked with secondary is greater hence induced emf is greater and the transformer is sep-up, similarly if the number of turns in the secondary coil are less than the number of turns in primary then total flux linked with secondary is lesser hence induced emf is smaller and the transformer is sep-down.

Theory :

The ratio of emf across secondary and emf across primary

$$\frac{\mathcal{E}_s}{\mathcal{E}_p} = \frac{-M\dfrac{di_p}{dt}}{-L\dfrac{di_p}{dt}} = \frac{M}{L}$$

$$\Rightarrow \quad \frac{\mathcal{E}_s}{\mathcal{E}_p} = \frac{\dfrac{\mu_o N_p N_s A}{\ell}}{\dfrac{\mu_o N_p{}^2 A}{\ell}} = \frac{N_s}{N_p}$$

For ideal transformer input power and output power are same hence

$$\mathcal{E}_s i_s = \mathcal{E}_p i_p$$

$$\Rightarrow \quad \frac{\mathcal{E}_s}{\mathcal{E}_p} = \frac{N_s}{N_p} = \frac{i_p}{i_s}$$

In a non-ideal transformer the out put voltage follows turn ratio where as current is less than what it should be in an ideal transformer.

$$i_s < \frac{N_p i_p}{N_s}$$

Efficiency of a transformer is the ratio of output power to input power.

Energy losses in transformer :

Following are the main reasons for losses in a transformer

Copper loss due to resistance of the wire used in winding of primary as well as secondary.

Eddy current loss (iron loss) due to eddy current established in iron core because of changing magnetic flux.

Loss due to leakage of magnetic flux since some of the flux linked with primary is not linked with secondary.

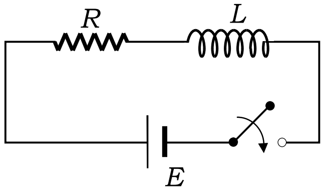
Hysteresis loss is due to periodic magnetization and demagnetization of the iron core.

Humming loss due to flow of alternating current through the coil it starts vibrating and there is loss of energy in the form of humming.

Uses :

It is mainly used to step up or step down the voltage as desired. They are also used in voltage regulators and stabilizers, large transformers are used to step up voltage at generating stations to feed supply lines.

$L - R$ circuit :



$$L\frac{di}{dt} + iR - E = 0$$

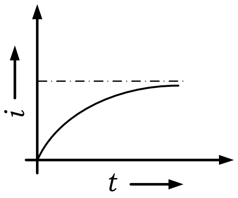
$$\Rightarrow \quad \int \frac{di}{E - iR} = \int \frac{dt}{L}$$

$$\Rightarrow \quad -\frac{1}{R}\ln(E - iR) = \frac{t}{L} + k$$

at $t = 0$, $i = 0$ thus

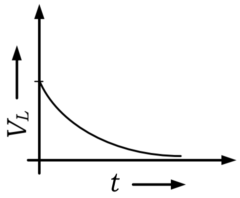
$$k = -\frac{1}{R}\ln(E)$$

$$\Rightarrow \quad i = \frac{E}{R}\left(1 - e^{-\frac{t}{(L/R)}}\right)$$



$$V_L = L\frac{di}{dt}$$

$$\Rightarrow \qquad V_L = E - iR = E\,e^{-\frac{t}{\left(L/R\right)}}$$



Moving Coil galvanometer :

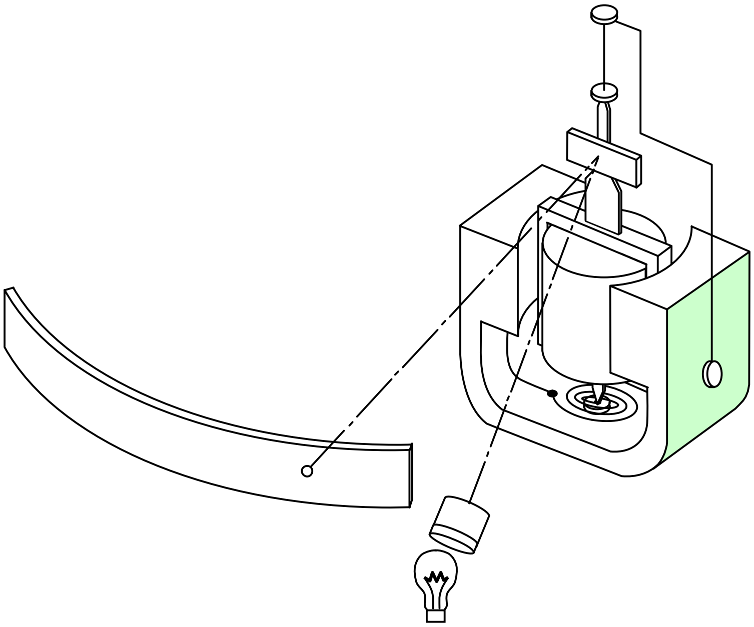
It is an instrument used to measure currents of small magnitudes flowing through an electrical circuit.

Principle :

A current carrying coil placed in a uniform magnetic field experiences a torque, which is proportional to current flowing through the coil.

Construction :

The galvanometer consists of a coil having large number of turns made of insulated wire and wound on a soft iron core of cylindrical shape. This assembly is suspended between the poles of a permanent magnet shaped cylindrical concave with the help of a thin torsion strip (usually phosphor bronze strip). The bottom end of axel of coil is placed on a pivot and one end of the coil is connected to a highly elastic heir spring at the bottom of the axel. The other end of the coil is connected to the upper phosphor bronze strip. A small mirror is attached on the strip for measuring deflection of coil with the help of lamp and scale arrangement. The entire arrangement is enclosed in a non magnetic enclosure to avoid external disturbances.



Working :

Concave pole pieces and cylindrical soft iron core produce a radial magnetic field in the gap in which coil rotates about vertical axis such that magnetic field always remains parallel to the plane of the coil, this makes torque on the coil due to magnetic field independent of its angle of deflection. Phosphor bronze strip and hair spring provide restoring torque to the coil and also form two terminals of the coil. When a current to be measured is passed through the coil gets deflected due to torque on it in magnetic field this produces torsion in the strip and hair spring which produces balancing torque for the coil. Deflection in the coil is measured with the help of lamp and scale arrangement in which a narrow light beam is focused on the mirror and reflected light spot is received on a translucent scale to measure deflection of coil.

Theory :

Torque on the coil due to magnetic field

$$\tau = niAB\sin\left(\pi/2\right) = niAB$$

as angle between axis of coil and magnetic field is always $\pi/2$. This torque is balanced by the restoring torque of the strip, therefore

$$niAB = C\theta$$

$$\Rightarrow \quad i = \frac{C}{nAB}\theta$$

$$\Rightarrow \quad i \propto \theta$$

Thus deflection produced in the coil is directly proportional to its angle of deflection.

Sensitivity of Galvanometer :

Current Sensitivity :

It is defined as the deflection of the galvanometer when one ampere current is passed through the galvanometer. The current sensitivity

$$S_I = \theta/I = \frac{nAB}{C}$$

Its units are rad/A.

To increase current sensitivity of the galvanometer, number of turns in the coil should be increased, magnetic poles should be made stronger (*B* should be increased), area of the coil should be increased or stiffness (C) of the strip as well as spring should be decreased.

Voltage Sensitivity :

It is defined as the deflection of the galvanometer when a potential difference of one volt is applied across it. The voltage sensitivity

$$S_V = \theta/V = \frac{\theta}{IR}$$

$$\Rightarrow \quad S_V = \frac{nAB}{CR}$$

Its units are rad/V.

To increase voltage sensitivity of the galvanometer, number of turns in the coil should be increased, magnetic poles should be made stronger (*B* should be increased), area of the coil should be increased, stiffness (C) of the strip as well as spring should be decreased or coil is made of a thicker wire (*R* should be decreased).