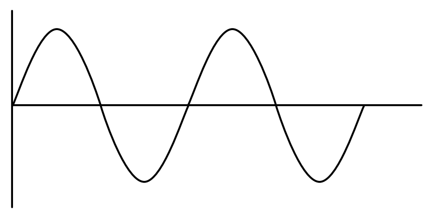
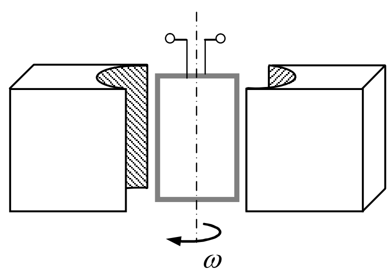
ALTERNATING CURRENT

Alternating Current :
Current flowing through an element changing its direction periodically is called an alternating current.



Alternating current can have very many types of wave shapes as a function of time, but here we are investigating ac with its magnitude changing sinusoidally with time.

ac Generation :
Magnetic flux is defined as



$$\Phi = \vec{\mathbf{B}}.\vec{\mathbf{S}} = BS\cos\theta$$

From farade's law of electromagnetic induction

$$\mathcal{E} = \frac{d\Phi}{dt}$$

For emf to be generated the flux must change with time, for which either of the quantities B, S or θ should change with time. Practically it is easer to change θ with time. Simply rotate the coil in a magnetic field with angular velocity ω. Then

$$\theta = \omega t$$

$\Rightarrow$ $$\Phi = BS\cos\omega t$$

$\Rightarrow$ $$\mathcal{E} = -\frac{d\Phi}{dt} = BS\omega\sin\omega t$$

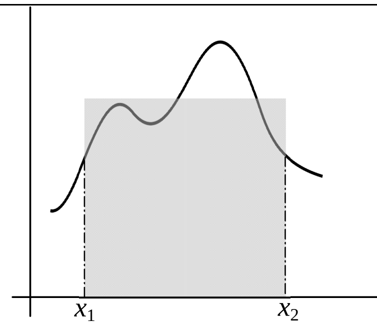
$\Rightarrow$ $$\mathcal{E} = \mathcal{E}_o\sin\omega t$$

This is alternating voltage. When this voltage is applied to a circuit the current of the form

$$i = i_o\sin(\omega t + \varphi)$$

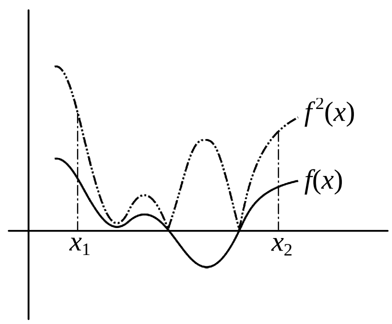
is established, called alternating current and φ the phase difference between voltage and current in the circuit.

Average and rms value of a function :
Mathematically the average of a function $y = f(x)$ in an interval $x_1 \rightarrow x_2$ is define as



$$\langle f(x)\rangle = \frac{\int_{x_1}^{x_2} f(x)\,dx}{\int_{x_1}^{x_2} dx} = \frac{1}{(x_2 - x_1)}\int_{x_1}^{x_2} f(x)\,dx$$

Graphically the average of a function is defined as the height of the rectangle on same base (x_1 to x_2) and of same area as that of area below the curve.
As the name suggests rms value of the function is



$$f(x)_{\text{rms}} = \sqrt{\frac{\int_{x_1}^{x_2} f^2(x)\,dx}{\int_{x_1}^{x_2} dx}} = \sqrt{\frac{1}{(x_2 - x_1)}\int_{x_1}^{x_2} f^2(x)\,dx}$$

When function goes negative its square still remains positive therefore the average of the square of the function is called mean square value of the function and its square root is the rms value.

Average current in complete time period : From definition of average value of the function

$$\langle i\rangle = \frac{\int_0^T i_o \sin\omega t\,dt}{\int_0^T dt}$$

$$\Rightarrow \quad \langle i\rangle = \frac{i_o}{\omega T}(-\cos\omega t)_0^T = 0$$

Average current in half time period :
From definition of average value of the function

$$\langle i\rangle = \frac{\int_0^{T/2} i_o \sin\omega t\, dt}{\int_0^{T/2} dt}$$

$$\Rightarrow \quad \langle i\rangle = \frac{2i_o}{\omega T}\left(-\cos\omega t\right)_0^{T/2}$$

$$\Rightarrow \quad \langle i\rangle = \frac{i_o}{\pi}\left(\cos 0 - \cos\frac{2\pi}{T}\frac{T}{2}\right) = \frac{2i_o}{\pi}$$

rms value of current in complete time period : From definition of average value of the function

$$i_{\text{rms}} = \sqrt{\frac{\int_0^T i_o^2 \sin^2\omega t\, dt}{\int_0^T dt}}$$

$$\Rightarrow \quad i_{\text{rms}} = \frac{i_o}{\sqrt{2T}}\sqrt{\int_0^T 2\sin^2\omega t\, dt}$$

$$\Rightarrow \quad i_{\text{rms}} = \frac{i_o}{\sqrt{2T}}\sqrt{\int_0^T \left(1-\cos 2\omega t\right) dt}$$

$$\Rightarrow \quad i_{\text{rms}} = \frac{i_o}{\sqrt{2T}}\sqrt{\left(t-\frac{\sin 2\omega t}{2\omega}\right)_0^T}$$

$$\Rightarrow \quad i_{\text{rms}} = \frac{i_o}{\sqrt{2}}$$

same will be the rms value of current for half of the cycle. rms value of current or voltage is also called effective value of current or voltage.

Simple ac circuits :
Since we are studying series circuits therefore we will consider current as the reference and try calculating voltage across the circuit element.

R-circuit :
Let

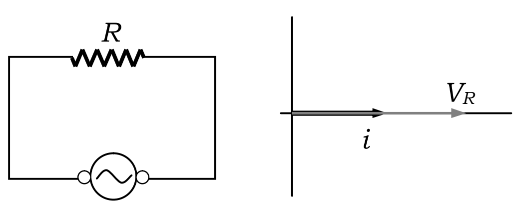
$$i = i_o \sin\omega t$$

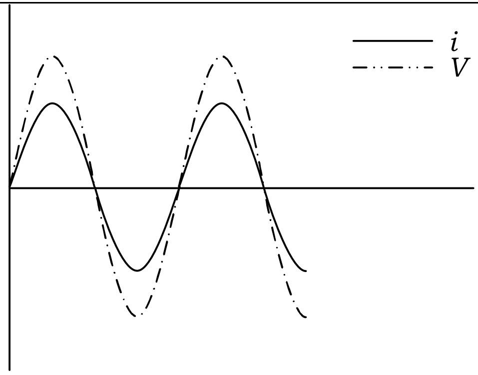
then voltage across resistance is

$$V = iR$$

$$\Rightarrow \quad V = \left(i_o \sin\omega t\right)R$$

$$\Rightarrow \quad V = V_o \sin\omega t$$





both voltage and current are in same phase.

C-circuit :
Let

$$i = i_o \sin \omega t$$

then voltage across capacitance is

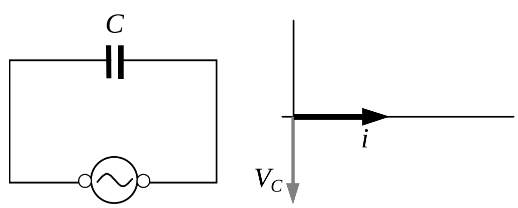
$$V = q/C$$

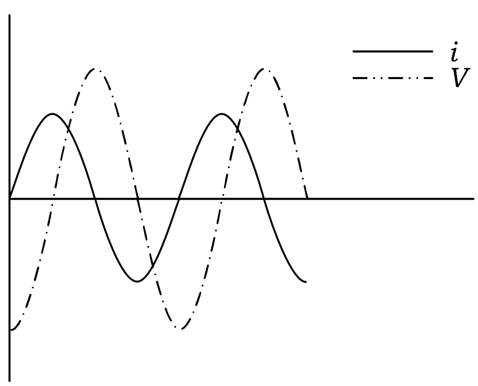
$\Rightarrow$ $$V = \frac{\int idt}{C}$$

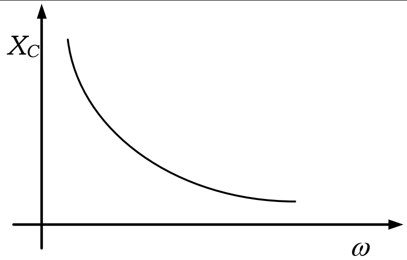
$\Rightarrow$ $$V = \left(i_o/C\right)\int \sin \omega t dt$$

$\Rightarrow$ $$V = \left(\frac{i_o}{\omega C}\right)(-\cos \omega t)$$

$\Rightarrow$ $$V = V_o \sin\left(\omega t - \pi/2\right)$$







voltage lags behind the current by a phase angle $\pi/2$. Now

$$V_o = \frac{i_o}{\omega C} = i_o X_C$$

form the above expression it is clear that the term $\left(1/\omega C\right)$ has the dimensions of resistance and called capacitive reactance. As X_C is inversely proportional to ω thus as frequency of ac increases X_C decreases. It is also a phasor quantity and lags behind the current by a phase angle $\pi/2$.

L-circuit :

Let

$$i = i_o \sin \omega t$$

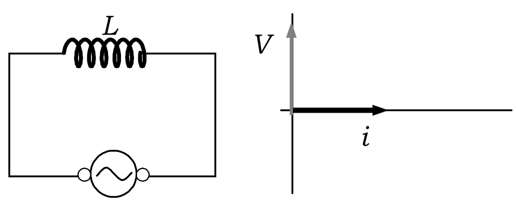
then voltage across inductance is

$$V = L\frac{di}{dt}$$

$\Rightarrow$ $$V = L\frac{d}{dt}\left(i_o \sin \omega t\right)$$

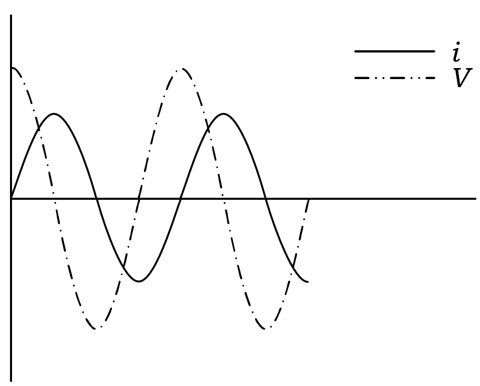
$\Rightarrow$ $$V = i_0 \omega L \cos \omega t$$

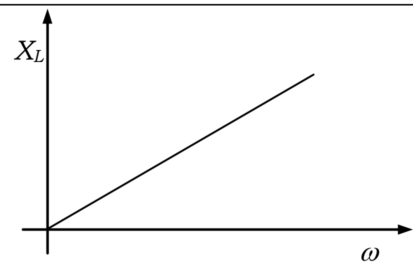
$\Rightarrow$ $$V = V_o \sin\left(\omega t + \pi/2\right)$$



voltage leads the current by a phase angle $\pi/2$. Now

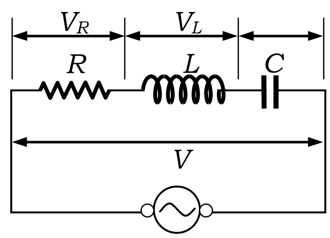
$$V_o = i_o \omega L = i_o X_L$$





form the above expression it is clear that the term (ωL) has the dimensions of resistance and called inductive reactance. As X_L is directly proportional to ω thus as the frequency of ac increases, X_L also increases linearly with ω. It is also a phasor quantity and leads the current by a phase angle $\pi/2$.

RLC-circuit :



At a particular instant

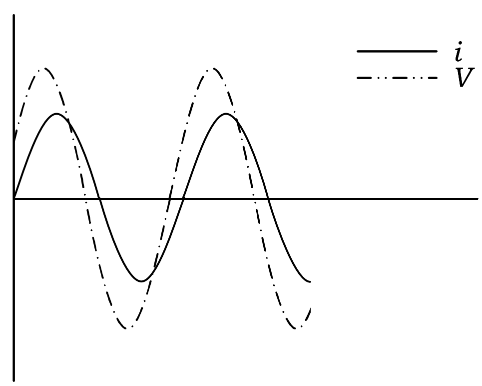
$$V_R = V_{R_0}\sin(\omega t) = i_0 R\sin(\omega t)$$

$$V_L = V_{L_0}\sin\left(\omega t + \pi/2\right) = i_0 X_L\sin\left(\omega t + \pi/2\right)$$

$$V_C = V_{C_0}\sin\left(\omega t - \pi/2\right) = i_0 X_C\sin\left(\omega t - \pi/2\right)$$

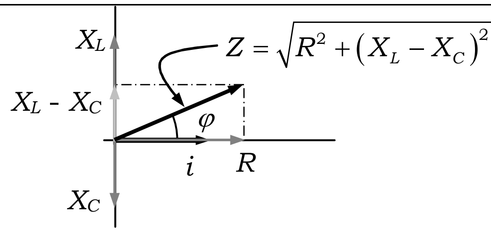
$$V = V_R + V_L + V_C$$

$\Rightarrow$ $$V = i_0 R\sin(\omega t) + i_0 X_L\sin\left(\omega t + \pi/2\right) + i_0 X_C\sin\left(\omega t - \pi/2\right)$$



$\Rightarrow$ $$V = i_0 R\sin(\omega t) + i_0(X_L - X_C)\cos(\omega t)$$

$\Rightarrow$ $$V = i_0\sqrt{R^2 + (X_L - X_C)^2}\left\{\frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}\sin(\omega t) + \frac{(X_L - X_C)}{\sqrt{R^2 + (X_L - X_C)^2}}\cos(\omega t)\right\}$$



$$\sin\varphi = \frac{X_L - X_C}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{X_L - X_C}{Z} \text{ and}$$

$$\cos\varphi = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{R}{Z}$$

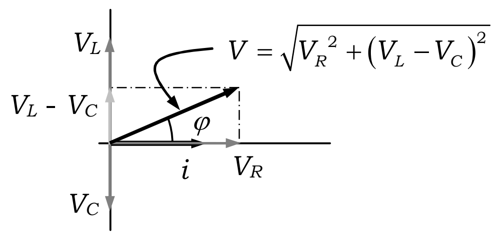
$\Rightarrow$ $V = i_0 Z\{\cos\varphi \sin(\omega t) + \sin\varphi \cos(\omega t)\}$

$\Rightarrow$ $V = i_0 Z \sin(\omega t + \varphi)$

Therefore in series circuit current flowing through all the elements remains same, hence

$V_0 = i_0 Z$, $V_{R_0} = i_0 R$, $V_{L_0} = i_0 X_L$ and $V_{C_0} = i_0 X_C$

$\Rightarrow$ $V = i_{rms} Z$, $V_R = i_{rms} R$, $V_L = i_{rms} X_L$ and $V_C = i_{rms} X_C$



In ac circuit voltages and currents are phasor quantities. Across resistance voltage phasor is in phase with the current, across inductor it 90° leading the current and across capacitor it is 90° lagging behind the current. Therefore from phasor diagram, the resultant voltage across the circuit is

$$V = \sqrt{V_R^2 + (V_L - V_C)^2}$$

This resultant (applied) voltage maintains a phase angle φ with the current in the circuit. From phasor diagram it is clear that

$$\tan\varphi = \frac{V_L - V_C}{V_R} \text{ or } \cos\varphi = \frac{V_R}{V}$$

Substituting values of V_R, V_L and V_C

$$V = \sqrt{(i_{rms} R)^2 + (i_{rms} X_L - i_{rms} X_C)^2}$$

$\Rightarrow$ $\dfrac{V}{i_{rms}} = \sqrt{R^2 + (X_L - X_C)^2}$

The quantity $\left(V/i_{rms}\right)$ has dimensions of resistance and called impedance of circuit. Thus

$$Z = \frac{V}{i_{rms}} = \sqrt{R^2 + (X_L - X_C)^2}$$

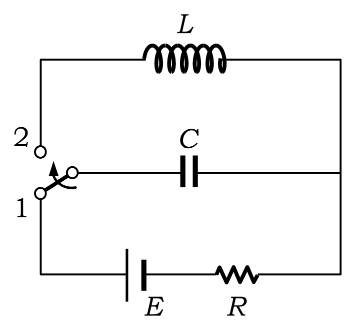
In ac circuit R, X_L, X_C, Z are also phasor quantities. The impedance of circuit maintains a phase angle φ with resistance (current in the circuit). From phasor diagram

$$\tan\varphi = \frac{X_L - X_C}{R} \text{ or}$$

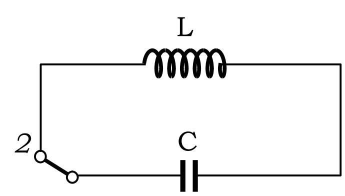
$$\cos\varphi = \frac{R}{Z}$$

LC oscillation :

When a charged capacitor is discharged through an inductor with negligible resistance in the circuit the current starts oscillating in the circuit.



Initially capacitor is charged keeping switch at position 1. It is thrown to position 2 when capacitor is charged. The capacitor starts discharging through inductor and potential energy stored in the form of electric field inside the capacitor starts decreasing where as the inductor gathers potential energy in the form magnetic potential energy. When capacitor is fully discharged (After quarter cycle of oscillation) the current through inductor is maximum and the entire energy of capacitor shows up as magnetic potential energy of the inductor. In the next quarter cycle this current charges capacitor with opposite polarity and once again the energy appears as electric potential energy of the capacitor. In the next half cycle the process repeated in opposite direction and one oscillation of current in the circuit is completed.



From Kirchoff's loop law, at any instant

$$\frac{q}{C} + L\frac{di}{dt} = 0$$

$$\Rightarrow \quad \frac{d^2i}{dt^2} + \frac{1}{LC}\frac{dq}{dt} = 0$$

$$\Rightarrow \quad \frac{d^2i}{dt^2} + \frac{1}{LC}i = 0$$

It is a second order differential equation very similar to differential equation of SHM. The solution of equation is

$$i = i_0 \sin(\omega t)$$

Where

$$\omega = \frac{1}{\sqrt{LC}}$$

And i_0 is the maximum current in the loop. Which can be calculated with the help of conservation of energy, as

$$\frac{1}{2}CE^2 = \frac{1}{2}Li_0^2$$

$$\Rightarrow \quad i_0 = E\sqrt{C/L}$$

Complex method of determining impedance of ac circuit :
In an ac circuit phasor of inductive reactance is 90° ahead of resistance where as phasor of capacitive reactance 90° lags behind the resistance. Therefore on an organ diagram resistance is plotted on real axis where as reactance on imaginary axis (inductive reactance on +ve imaginary axis and capacitive reactance on –ve imaginary axis). For this ω is replaced by $j\omega$ where

$j = \sqrt{-1}$, hence

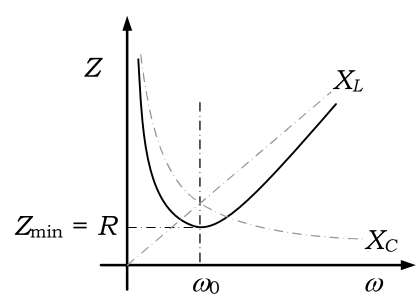
$X_L = j\omega L$ and

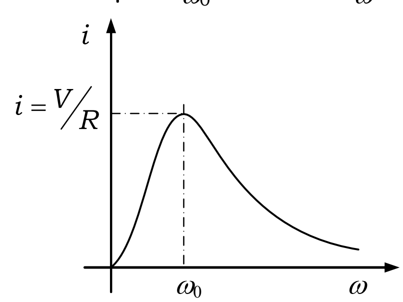
$$X_C = \frac{1}{j\omega C} = -\frac{j}{\omega C}$$

Since set of complex number is closed for all mathematical operations therefore after writing the complex equivalent of all reactances, the circuit is solved for equivalent resistance following simple rules of series, parallel combination etc. We expect to get a complex number Z as the equivalent resistance in the complex form. Magnitude of Z gives total impedance of the circuit.

Resonance in an ac circuit :
An ac circuit is said to be resonant when the reactive part of the impendence becomes zero. This happens at a particular frequency of ac called resonant frequency of the circuit.

Resonance in series circuit :
As the total impedance Z of the circuit depends on the frequency ω of ac in the circuit. X_L increases with ω where as X_C decreases, hence the magnitude of $(X_L - X_C)$ is minimum when $X_L = X_C$, at this point in series ac circuit Z is also minimum ($Z_{min} = R$). At this situation series circuit is said to be resonant. Hence for resonance





$X_L = X_C$

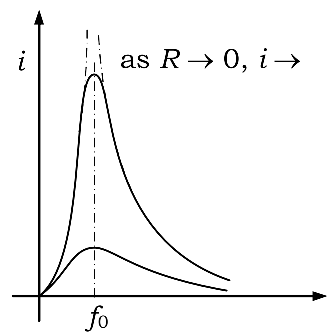
$\Rightarrow \quad \omega_0 L = \frac{1}{\omega_0 C}$

$\Rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}} = 2\pi f_0$

$\Rightarrow \quad f_0 = \frac{1}{2\pi\sqrt{LC}}$

The series resonant circuit is called acceptor circuit because the impedance of the circuit is minimum at resonance so that it allows the current of the resonant frequency out of currents of many frequencies to pass through the circuit.

Q factor of series ac resonant circuit (sharpness of resonance):
At resonance in series ac circuit $X_L = X_C$ hence the voltage across them is also equal in magnitude (but opposite in phase). Therefore voltage across inductor (or capacitor) is



$V_L = i_{rms} X_L$

$\Rightarrow \quad V_L = \left(\frac{V_{rms}}{Z}\right)\omega L = \left(\frac{V_{rms}}{R}\right)\omega L$ *

The ratio of voltage across inductor (or capacitor as they are equal) and resultant voltage (applied voltage) is defined as the Q factor of circuit.

$Q = \frac{V_L}{V} = \frac{\omega_0 L}{R}$

Q is a dimensionless number and determines the sharpness of the resonance.

At resonance $Z = R$ hence, the power consumption by the circuit is maximum as Z is minimum. If the frequency at which half of this maximum power is consumed, is

$(\omega_0 + \Delta\omega)$

then current at this frequency must be $1/\sqrt{2}$ times the current at the resonance, therefore

$1/\sqrt{2}\left(\frac{V}{R}\right) = \frac{V}{Z}$

$\Rightarrow \quad 2R^2 = R^2 + (X_L - X_C)^2$

* $i_{rms} = \frac{V_{rms}}{Z}$ at resonance $Z = R$, hence at resonance $i_{rms} = \frac{V}{R}$

$$\Rightarrow \quad R = (\omega_0 + \Delta\omega)L - \frac{1}{(\omega_0 + \Delta\omega)C}$$

$$\Rightarrow \quad R = (\omega_0 + \Delta\omega)L - \frac{\omega_0^2 L}{(\omega_0 + \Delta\omega)}$$

$$\Rightarrow \quad R = \omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0}\right) - \frac{\omega_0 L}{\left(1 + \frac{\Delta\omega}{\omega_0}\right)}$$

As $\Delta\omega$ is very small compared to resonance frequency ω_0, hence

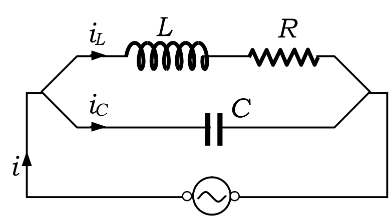
$$\Rightarrow \quad R \approx \omega_0 L\left(1 + \frac{\Delta\omega}{\omega_0}\right) - \omega_0 L\left(1 - \frac{\Delta\omega}{\omega_0}\right)$$

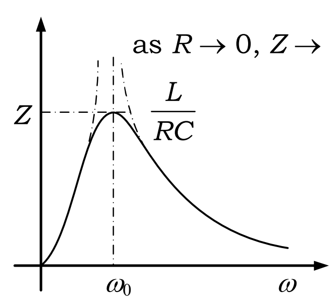
$$\Rightarrow \quad R \approx 2\Delta\omega L$$

$$\Rightarrow \quad \frac{\omega_0}{2\Delta\omega} \approx \frac{\omega_0 L}{R}$$

The frequency range $(\omega_0 - \Delta\omega)$ to $(\omega_0 + \Delta\omega)$ is defined as the bandwidth of the resonant circuit, and smaller is the bandwidth greater is the sharpness of resonance.

Parallel resonance circuit :
A circuit is said to be resonant when reactive part of the impedance of the circuit is zero. Hence calculating total impedance of the circuit in complex form





$$Z = \frac{(R + j\omega L)\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}$$

$$\Rightarrow \quad Z = \frac{(R + j\omega L)}{(R + j\omega L)j\omega C + 1}$$

$$\Rightarrow \quad Z = \frac{(R + j\omega L)}{(1 - \omega^2 LC) + j\omega RC}$$

on rationalization of denominator gives

$$Z = \frac{\left\{R\left(1-\omega^2LC\right)+\omega^2LRC\right\}+j\omega\left\{L\left(1-\omega^2LC\right)-R^2C\right\}}{\left(1-\omega^2LC\right)^2+\omega^2R^2C^2}$$

$\Rightarrow$ $$Z = \frac{R+j\omega\left\{L\left(1-\omega^2LC\right)-R^2C\right\}}{\left(1-\omega^2LC\right)^2+\omega^2R^2C^2}$$

for resonance, putting reactive part of the impedance equal to zero, we get

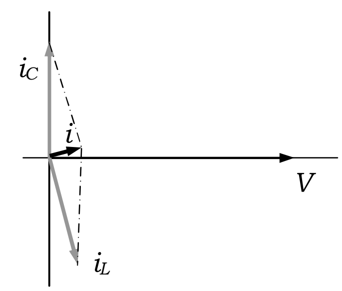
$$L\left(1-\omega_0{}^2LC\right)-R^2C=0$$

$\Rightarrow$ $$\omega_0 = \sqrt{\frac{1}{LC}-\frac{R^2}{L^2}}$$

at resonance the impedance is resistive, putting the value of ω_0 in the above expression of Z we get

$$Z = \frac{L}{RC}$$

hence at resonance as if R is small, Z is large.



For resonance to occur the ω_0 should be real therefore

$$\frac{1}{LC}-\frac{R^2}{L^2}>0$$

$\Rightarrow$ $$R<\sqrt{L/C}$$

it is the necessary condition for resonance in parallel ac circuit.

If we take voltage as the reference in parallel ac circuit, the current through inductor (i_L) is lagging in phase compared to voltage across it and the current through capacitor (i_C) is leading voltage by 90°.
The parallel resonant circuit is called rejecter circuit (or filter circuit) because the impedance of the circuit is maximum at resonance so that it blocks the current of the resonant frequency out of currents of many frequencies from passing through the circuit.

<u>Average power in an ac circuit</u> :
Let the instantaneous values of current and voltage in an ac circuit be

$$i = i_o \sin\omega t \quad \text{and}$$

$$V = V_o \sin\left(\omega t+\varphi\right)$$

hence the instantaneous power in the circuit is

$$P = iV = i_oV_o\sin\left(\omega t\right)\sin\left(\omega t+\varphi\right)$$

therefore the average power in the circuit is

$$\langle P\rangle = \frac{i_o V_o}{2}\frac{\int_0^T 2\sin(\omega t)\sin(\omega t+\varphi)dt}{\int_0^T dt}$$

$$\Rightarrow \quad \langle P\rangle = \frac{i_o V_o}{2T}\int_0^T \{\cos(\varphi)-\cos(2\omega t+\varphi)\}dt$$

$$\Rightarrow \quad \langle P\rangle = \frac{i_o V_o}{2T}\left\{\cos(\varphi)t-\frac{\sin(2\omega t+\varphi)}{2\omega}\right\}_0^T$$

$$\Rightarrow \quad \langle P\rangle = \frac{i_o V_o}{2}\cos(\varphi)$$

$$\Rightarrow \quad \langle P\rangle = i_{rms}V_{rms}\cos(\varphi)$$

therefore if $\varphi = \pi/2$ (in pure inductive or capacitive circuit) the average power consumption is zero even when current is flowing through it, such current is called wattles current.
The average power in ac circuit can also be written as

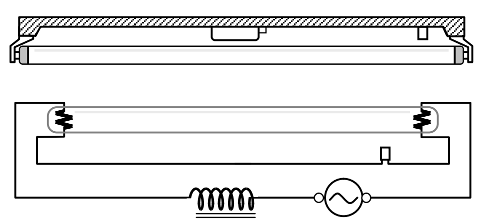
$$\langle P\rangle = i_{rms}V_{rms}\frac{R}{Z}$$

$$\Rightarrow \quad \langle P\rangle = i_{rms}^{\,2}R$$

$$\Rightarrow \quad \langle P\rangle = V_{rms}^{\,2}\frac{R}{Z^2}$$

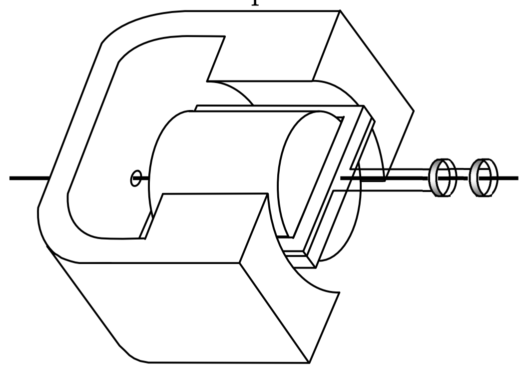
Watt-less current :

Choke coil :
It is a simple inductor coil of large self inductance (hence large inductive reactance X_L) and small resistance. In an ac circuit to reduce current without appreciable loss of power this coil is put in series with the circuit element. For example in house hold tube lights, for the successful operation lesser amount of current is needed when tube is in operation. Putting a resistance in series or an inductor/capacitor can either do this. But resistance produces heat and consumes power where as inductor or capacitor reduces current but does not consume appreciable power.



ac Generator :
An ac generator is a machine which converts mechanical energy in to electrical energy and produces alternating current.
Principle :
This device works on the principle of electromagnetic induction. When a coil is rotated in a magnetic field, flux linked with it changes continuously and generates induced emf in the coil.
Construction :
Main parts of an ac generator are
Armature : A rectangular coil *ABCD* consisting of large number of turns made of insulated copper wire wound over a soft iron core.

Field Magnet : Armature coil is placed between poles of a strong permanent magnet made concave for the maximum linkage of magnetic flux with the armature coil.
Slip Rings : Two metallic rings connected to the two ends of the coil are so connected that they provide sliding contacts to the output terminals with the help of brushes.



Working :
Armature coil is made to rotate between the poles of the permanent magnet (field magnets). The angle between magnetic field and the normal to the coil changes with time because of which flux liked with the coil changes with time and the emf is induced in the coil which changes sinusoidaly. According to Lenz's law the direction of induced emf is such that it opposes the rotation of the coil hence mechanical energy is converted to electrical energy.
Theory :
Total magnetic field bound by the armature coil at any instant is

$$\varphi_B = N\,\vec{\mathbf{S}}.\vec{\mathbf{B}}$$

$$\Rightarrow \quad \varphi_B = N\,BS\cos\theta = N\,BS\cos(\omega t)$$

Induced emf in the coil

$$\mathcal{E} = -\frac{d}{dt}\varphi_B$$

$$\Rightarrow \quad \mathcal{E} = -\frac{d}{dt}N\,BS\cos(\omega t)$$

$$\Rightarrow \quad \mathcal{E} = NBS\omega\sin(\omega t)$$

$$\Rightarrow \quad \mathcal{E} = \mathcal{E}_0\sin(\omega t)$$

This sinusoidaly varying voltage alternates twice in a cycle. Hence it is called an ac generator (more precisely an ac alternator).